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Characterizing all models in infinite cardinalities

Lauri Keskinen *

FNWI, ILLC, Universiteit van Amsterdam, P.O. Box 94242, 1090 GE Amsterdam, The Netherlands

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ABSTRACT

Fix a cardinal κ. We can ask the question: what kind of a logic L is needed to characterize all models of cardinality κ (in a finite vocabulary) up to isomorphism by their L-theories? In other words: for which logics L it is true that if any models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ satisfy the same L-theory then they are isomorphic?
It is always possible to characterize models of cardinality κ by their L_{κ^+,κ^+}-theories, but we are interested in finding a "small" logic L, i.e., the sentences of L are hereditarily of smaller cardinality than κ. For any cardinal κ it is independent of *ZFC* whether any such small definable logic L exists. If it exists it can be second order logic for $\kappa = \omega$ and fourth order logic or certain infinitary second order logic $L^2_{\kappa,\omega}$ for uncountable κ. All models of cardinality κ can always be characterized by their theories in a small logic with generalized quantifiers, but the logic may be not definable in the language of set theory. Our work continues and extends the work of Ajtai [Miklos Ajtai, Isomorphism and higher order equivalence, Ann. Math. Logic 16 (1979) 181–203].

1. Introduction

We shall investigate whether second order equivalence of two models, or equivalence in some stronger logic than second order logic, implies isomorphism of the models in certain cardinalities. We always assume that our vocabulary is finite. The notation which is not yet explained can be found under the heading "Notation" below.

Remark 1.1. We are assuming throughout this paper that the vocabulary is finite. This is because if the vocabulary is finite, then the isomorphism type of the model is characterizable inside the model in second order logic. In infinitary second order logic $L^2_{\kappa,\omega}$ the isomorphism type of the model is characterizable if the vocabulary is smaller than κ, and our assumption is stronger than what is needed.

The following lemma of Shelah demonstrates that not all countable models with countable vocabularies can be characterized by their second order theories.

Lemma 1.2 *(Shelah). There are two countable non-isomorphic second order equivalent models in a countably infinite vocabulary. The models are also L^n-equivalent for any n.*

Proof. The vocabulary of the models contains infinitely many constants $\{c_n\colon n \in \omega\}$. Let $\mathfrak{A}$ be a model such that $\mathrm{dom}(\mathfrak{A}) = \{a_n\colon n \in \omega\}$ and $c_n^{\mathfrak{A}} = a_n$ for each n. Let $\mathrm{dom}(\mathfrak{B}) = \{a_n\colon n \in \omega\} \cup \{a_\omega\}$ and $c_n^{\mathfrak{B}} = a_n$ for each n.

* Correspondence address: Palsinantie 376 A, 35820 Mänttä, Finland. Tel.: +358465608580.
E-mail address: laurikeskinen1@gmail.com.

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The models are not isomorphic as in the model $\mathfrak{A}$ every element is an interpretation of some constant but in $\mathfrak{B}$ the element a_ω is not an interpretation of any constant. We claim that the models $\mathfrak{A}$ and $\mathfrak{B}$ are L^n-equivalent for any n. So take an arbitrary L^n-sentence ϕ. Let τ be the finite set of constants in ϕ. Now $\mathfrak{A} \restriction \tau$ is isomorphic to $\mathfrak{B} \restriction \tau$ and it follows that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same L^n-sentences in the vocabulary τ. Thus $\mathfrak{A} \models \phi \Leftrightarrow \mathfrak{B} \models \phi$. □

Suppose L is a logic [3] (Chapter 2, Definition 1.1.1). The *L-theory* of a model is the set of L-sentences true in the model. Two models are said to satisfy the same L-theory if they satisfy the same L-sentences.

Definition 1.3. We use the expression $A(L, \kappa)$ to refer to the following condition: For any models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ, if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same L-theory then they are isomorphic.

We use $A(ZF, \kappa)$ to denote the condition "for all models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ in a finite vocabulary, if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences (with the model as a parameter) in the language of set theory then $\mathfrak{A} \cong \mathfrak{B}$." Note that *ZF* is not a logic as two isomorphic models can satisfy different sentences in the language of set theory.

Definition 1.4. We call $A(L^2, \omega)$ when restricted to ordinals the *Fraïssé Hypothesis*. This is the Hypothesis: All countable ordinals have different second order theories.

Ajtai [2] has proved that $A(L^2, \omega)$ is independent of *ZFC*. We are looking for related results in the cardinality $\aleph_0$ and similar results in higher cardinalities. The name "Fraïssé Hypothesis" has been introduced by Wiktor Marek. The Fraïssé Hypothesis has been studied by Fraïssé [4] and Marek [12,13].

Our results are relative to the consistency of *ZFC*. If we assume more than the consistency of *ZFC* it is always explicitly mentioned.

In Section 3 we will recall the proof of Ajtai and make some observations related to $A(L^2, \omega)$.

In Section 4 we will develop a forcing technique for coding subsets of ordinals by collapsing certain cardinals. This forcing is used to prove for example the following: If κ is a cardinal in L, then there is a transitive model of *ZFC* in which $A(L^4, \lambda)$ holds for exactly cardinals λ smaller than or equal to κ.

In Section 5 we will show that if κ is a cardinal, then there is a language $L^{\kappa *}$ with κ many generalized quantifiers such that $A(L^{\kappa *}, \kappa)$ holds. Given a cardinal κ the language $L^{\kappa *}$ may be different for different models of *ZFC* containing κ and it is also possible that no such $L^{\kappa *}$ is definable in the language of set theory. This result for $\kappa = \omega$ is due to Scott Weinstein (Personal communication with Jouko Väänänen) and the generalization for uncountable κ is based on an idea of Per Lindström (Personal letter to Jouko Väänänen, 1 August 1974).

In Section 6 we will use Ajtai's method to prove that it is independent of *ZFC* whether $A(L^2_{\kappa,\omega}, \kappa)$ holds for a regular cardinal κ. We will also prove that for different regular cardinals κ and λ, $A(L^2_{\kappa,\omega}, \kappa)$ and $A(L^2_{\lambda,\omega}, \lambda)$ are independent of each other. We will also give an analogous result for singular cardinals.

In Section 7 we will investigate the relation between $A(L^2, \omega)$ and various large cardinal axioms. If there are infinitely many Woodin cardinals and a measurable cardinal above them, then $A(L^2, \omega)$ fails. Assuming the consistency of relevant large cardinal axioms, if n is a natural number, then there is a model of *ZFC* in which there are n Woodin cardinals and $A(L^2, \omega)$ holds. As n grows bigger, more complex second order sentences seem to be needed to characterize all countable models up to isomorphism. $A(L^3, \omega)$ is consistent with Martin's Maximum and practically all large cardinal axioms.

For a discussion of the role of second order characterizations in the foundations of mathematics see [21].

Notation

The expression *ZF-formulas* refers to formulas in the language of set theory, i.e., first order language in a vocabulary with one binary relation $\in$. *ZF-equivalence* of two structures, denoted by $\mathfrak{A} \equiv_{ZF} \mathfrak{B}$, refers to the condition that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same formulas of the language of set theory, i.e., for any formula $\phi(x)$ in the language of set theory $V \models \phi(\mathfrak{A}) \leftrightarrow \phi(\mathfrak{B})$. If L is a logic $\mathfrak{A} \equiv_L \mathfrak{B}$ refers to the condition that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of L. The expression $H(\kappa)$ refers to the set of sets hereditarily smaller than κ, i.e., $\{X$: the transitive closure of X has cardinality less than $\kappa\}$. The symbol $\restriction$ means "restricted to". Depending on context this can mean a reduct of a model to a smaller vocabulary or restriction of some operations to some set. The notation $\phi^{\mathfrak{M}}(\cdot)$ refers to the set of tuples which satisfy the formula ϕ in model $\mathfrak{M}$. A forcing name of a given set X is denoted by $\dot{X}$. Interpretation of a definable set in a given model of *ZFC* is denoted by the set with the model of *ZFC* as superscript: for example ω_1^{L} means the ω_1 of L. If t is a term, $\mathfrak{A}$ is a model and s is an assignment which maps the free variables of t to elements of $\mathfrak{A}$, $t_s^{\mathfrak{A}}$ refers to the interpretation of t in $\mathfrak{A}$ with assignment s. Analogously if R is a higher order variable $R_s^{\mathfrak{A}}$ refers to the interpretation of the higher order variable R in the model $\mathfrak{A}$ with assignment s. If the higher order variable has subscripts and superscripts, we use parentheses for clarity: for example $(R_j^i)_s^{\mathfrak{A}}$. The expression $\mathfrak{A} \models_s \phi$ refers to the condition that the formula ϕ with the assignment s is satisfied in the model $\mathfrak{A}$. By *the reals* we mean the power set of ω.

Notation which is not explained is standard as used for example in Jech's book [9].