Annals of Pure and Applied Logic 164 (2013) 284–293

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Reverse mathematics and Peano categoricity

Stephen G. Simpson [a,*], Keita Yokoyama [b,1]

[a] Department of Mathematics, Pennsylvania State University, United States
[b] Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Japan

ARTICLE INFO

Article history:
Received 28 March 2012
Received in revised form 28 September 2012
Accepted 9 October 2012
Available online 27 October 2012
Communicated by U. Kohlenbach

MSC:
primary 03B30
secondary 03D80, 03F35, 00A30

Keywords:
Reverse mathematics
Second-order arithmetic
Peano system
Foundations of mathematics
Proof theory
Second-order logic

ABSTRACT

We investigate the reverse-mathematical status of several theorems to the effect that the natural number system is second-order categorical. One of our results is as follows. Define a *system* to be a triple A, i, f such that A is a set and $i \in A$ and $f : A \to A$. A subset $X \subseteq A$ is said to be *inductive* if $i \in X$ and $\forall a (a \in X \Rightarrow f(a) \in X)$. The system A, i, f is said to be *inductive* if the only inductive subset of A is A itself. Define a *Peano system* to be an inductive system such that f is one-to-one and $i \notin$ the range of f. The standard example of a Peano system is $\mathbb{N}, 0, S$ where $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ = the set of natural numbers and $S : \mathbb{N} \to \mathbb{N}$ is given by $S(n) = n + 1$ for all $n \in \mathbb{N}$. Consider the statement that all Peano systems are isomorphic to $\mathbb{N}, 0, S$. We prove that this statement is logically equivalent to WKL_0 over RCA_0^*. From this and similar equivalences we draw some foundational/philosophical consequences.

1. Introduction

Reverse mathematics[2] is a well known [18,20] research direction in the foundations of mathematics. The goal of reverse mathematics is to pinpoint the weakest set-existence axioms which are needed in order to prove specific theorems of core mathematics. Such investigations are most fruitfully carried out in the context of subsystems of second-order arithmetic [18]. In that context it frequently happens that the weakest axioms needed to prove a particular theorem are logically equivalent to the theorem, over a weak base theory. For example, the well known theorem that every uncountable closed set in Euclidean space contains a perfect subset is logically equivalent to ATR_0 over the weak base theory RCA_0 [18, Theorem V.5.5].

A key theorem in rigorous core mathematics is the categoricity of the natural number system. Stated more precisely and in 20th-century language, the *Peano Categoricity Theorem* [14, Theorem 2.7.1] asserts that any two Peano systems are isomorphic. The Peano Categoricity Theorem was originally proved by Dedekind in 1888 ([4, Satz 132], [5, Theorem 132])

* Corresponding author.
E-mail addresses: simpson@math.psu.edu (S.G. Simpson), yokoyama.k.ai@m.titech.ac.jp (K. Yokoyama).
URL: http://www.math.psu.edu/simpson (S.G. Simpson).

1 Yokoyama's research was supported by a Japan Society for the Promotion of Science postdoctoral fellowship for young scientists, and by a grant from the John Templeton Foundation.

2 A crucial role in the development of reverse mathematics was played by H. Friedman [7,8]. We thank the referee for strongly suggesting that we include this historical comment.

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2012.10.014

as a highlight of his rigorous, set-theoretical development [3–5] of the natural number system $\mathbb{N}$ and the real number system $\mathbb{R}$.

In this paper we investigate the reverse-mathematical and proof-theoretical status of the Peano Categoricity Theorem and related theorems. One of our results is as follows.

$$\text{The Peano Categoricity Theorem is equivalent to } \mathsf{WKL}_0 \text{ over the standard weak base theory } \mathsf{RCA}_0. \tag{1}$$

Here RCA_0 and WKL_0 are familiar [18,20] subsystems of second-order arithmetic. Namely, RCA_0 consists of Δ^0_1 comprehension plus Σ^0_1 induction, and WKL_0 consists of RCA_0 plus Weak König's Lemma.

Our result (1) offers further confirmation of a point made by Väänänen[3] in a recent talk based on his recent paper [22]. Väänanen observed that various second-order categoricity theorems can be proved without resorting to the full strength of second-order logic. Clearly (1) bears this out, because WKL_0 is a relatively weak[4] subsystem of second-order arithmetic, much weaker than ACA_0 and in fact Π^0_2-equivalent to Primitive Recursive Arithmetic [18, §IX.3]. Since by (1) the Peano Categoricity Theorem is provable in WKL_0, it follows that the Peano Categoricity Theorem is *finitistically reducible* in the sense of Simpson's partial realization [17,19] (see also [1]) of Hilbert's Program [9].

As a refinement of (1) we obtain the following stronger result.

$$\text{The Peano Categoricity Theorem is equivalent to } \mathsf{WKL}_0 \text{ not only over } \mathsf{RCA}_0 \text{ but over the much weaker base theory } \mathsf{RCA}^*_0. \tag{2}$$

Recall from Simpson [18, §X.4] and Simpson and Smith [21] that RCA^*_0 is RCA_0 with Σ^0_1 induction weakened to Δ^0_1 induction plus *natural number exponentiation*, i.e., the assertion that m^n exists for all $m, n \in \mathbb{N}$. It is known that RCA^*_0 is Π^0_2-equivalent to Elementary Function Arithmetic [21], hence much weaker than RCA_0 and WKL_0 which are Π^0_2-equivalent to Primitive Recursive Arithmetic [18, §IX.3].

Our stronger result (2) provides some foundational or philosophical insight concerning Dedekind's construction of the natural number system [4,5]. Recall that Dedekind's key technical lemma, the "Satz der Definition durch Induction," is a straightforward embodiment[5] of the idea of primitive recursion. But at the same time, according to (2), the Peano Categoricity Theorem itself *requires* primitive recursion. Thus (2) constitutes further evidence that primitive recursion is indeed the heart of the matter.

The plan of this paper is as follows. In Section 2 we prove (1). In Section 3 we prove (2). In Section 4 we investigate the reverse-mathematical status of certain variants of the Peano Categoricity Theorem, replacing the Peano system $\mathbb{N}, 0, S$ by the ordered system $\mathbb{N}, 0, <$ or the ordered Peano system $\mathbb{N}, 0, <, S$. In Section 5 we summarize our results and state some open questions.

2. The role of Weak König's Lemma

Recall from [18] that RCA_0 is the subsystem of second-order arithmetic consisting of Δ^0_1 comprehension and Σ^0_1 induction. Within RCA_0 one may freely use primitive recursion and minimization to define functions $g : \mathbb{N}^k \to \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers [18, §II.3]. Recall also [18] that WKL_0 consists of RCA_0 plus Weak König's Lemma, i.e., the statement that every infinite tree $T \subseteq \{0,1\}^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} \{0,1\}^n$ has an infinite path.

The purpose of this section is to show that the Peano Categoricity Theorem is equivalent to Weak König's Lemma, the equivalence being provable in RCA_0.

Definition 2.1. Within RCA_0 we make the following definitions. A *system* is a triple A, i, f such that $A \subseteq \mathbb{N}$ and $i \in A$ and $f : A \to A$. A *Peano system* is a system A, i, f such that $i \notin \mathrm{rng}(f)$ and f is one-to-one and

$$(\forall X \subseteq A)\,\big(\big(i \in X \text{ and } \forall a\,\big(a \in X \Rightarrow f(a) \in X\big)\big) \Rightarrow X = A\big).$$

The standard example of a Peano system is $\mathbb{N}, 0, S$ with $S : \mathbb{N} \to \mathbb{N}$ defined by $S(n) = n + 1$. A Peano system A, i, f is said to be *isomorphic to* $\mathbb{N}$ if there exists a bijection $\Phi : A \to \mathbb{N}$ such that $\Phi(i) = 0$ and $\Phi(f(a)) = \Phi(a) + 1$ for all $a \in A$. A Peano system A, i, f is said to be *almost isomorphic to* $\mathbb{N}$ if for each $a \in A$ there exists $n \in \mathbb{N}$ such that $f^n(i) = a$.

Lemma 2.2. *The following is provable in* RCA_0. *If a Peano system is almost isomorphic to* $\mathbb{N}$, *it is isomorphic to* $\mathbb{N}$.

Proof. We reason in RCA_0. Let A, i, f be a system. As in [18, §II.3] use Σ^0_1 induction to prove that for all $n \in \mathbb{N}$, $f^n(i)$ exists and $f^n(i) \in A$. Use Δ^0_1 comprehension to prove the existence of the function $n \mapsto f^n(i) : \mathbb{N} \to A$. Assume now that

3 We thank Jouko Väänänen for raising the question which is answered by (1).

4 By the *strength* of a theory T we mean the set of Π^0_1 sentences which are provable in T.

5 Dedekind's "Satz der Definition durch Induction" ([4, Satz 126], [5, Theorem 126]) may be restated in 20th-century language [14, Theorem 2.2.1] as follows. For any system A, i, f there is a unique function $\Phi : \mathbb{N} \to A$ such that $\Phi(0) = i$ and $\Phi(n + 1) = f(\Phi(n))$ for all $n \in \mathbb{N}$.