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Algebraically closed MV-algebras and their sheaf representation

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ABSTRACT

In this paper we first provide a new axiomatization of algebraically closed MV-algebras based on McNaughton's Theorem. Then we turn to sheaves, and we represent algebraically closed MV-algebras as algebras of global sections of sheaves, where the stalks are divisible MV-chains and the base space is Stonean.

1. Introduction

In this paper we are interested in *MV-algebras*. These algebras are the many-valued equivalents of Boolean algebras, which are the algebraic semantics of Classical 2-valued Logic. It is well known that free MV-algebras over any set of generators are MV-algebras of McNaughton's functions, which are piecewise linear functions, so, it is natural to associate to free MV-algebras the geometry of simplexes [4]. However, simplexes could be no more sufficient when one passes to more general MV-algebras. So we would like to understand the algebraic geometry of general MV-algebras, in analogy with classical algebraic geometry over rings.

A basic tool of algebraic geometry consists of algebraically closed fields which provide a natural environment for algebraic varieties (that are sets of zeros of systems of polynomial equations). Another modern approach due to Grothendieck and Serre is based on the notions of sheaf and scheme. We would like to combine both tools in order to try to understand some algebraic geometry for MV-algebras. This paper is a first step in this direction.

Algebraically closed MV-algebras (a.c. MV-algebras) are studied by Lacava in [11] and [12]. In the latter paper he characterizes a.c. MV-algebras as regular and divisible MV-algebras. The main results of this paper are:

- an axiomatization of algebraically closed MV-algebras different from the one of [11];
- a sheaf representation of algebraically closed MV-algebras via divisible MV-chains and Stone spaces.

We would like to stress that the sheaf representation of a.c. MV-algebras is more concrete than the purely algebraic characterization by Lacava since it gives a decomposition of an a.c. MV-algebra into simpler and better known MV-algebras

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(remember that divisible MV-chains correspond via Mundici's functor Γ (see [4]) to totally ordered divisible abelian groups).

2. Preliminaries

The main tool we use in this paper is sheaf representation of algebras. We borrow from [5] the standard definition of a sheaf of sets or a sheaf of algebras belonging to a given variety. In particular, we have the notion of a sheaf of MV-algebras.

In order to prove the completeness theorem of Łukasiewicz infinite-valued logic, Chang introduced MV-algebras in [3]. In order to make this preliminary section not too long we give only a quick review of MV-algebras, referring to [4] for further details.

An *MV-algebra* is a structure $(A, \oplus, ^*, 0)$, where $\oplus$ is a binary operation, * is a unary operation and 0 is a constant such that the following axioms are satisfied for any a, $b \in A$:

i) $(A, \oplus, 0)$ is an abelian monoid,
ii) $(a^*)^* = a$,
iii) $0^* \oplus a = 0^*$,
iv) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

On an MV-algebra A we define the constant 1 and the auxiliary operation $\odot$ as follows:

v) $1 := 0^*$,
vi) $a \odot b := (a^* \oplus b^*)^*$

for any a, $b \in A$.

An example of an MV-algebra is given by the real interval $[0, 1]$ where $x \oplus y = \min\{x + y, 1\}$ and $x^* = 1 - x$. This MV-algebra is important because it generates the variety of all MV-algebras.

For the partial order on MV-algebras $x \leqslant y$ and the infimum and supremum $x \wedge y$, $x \vee y$ we refer to [4]. Also, for boolean (i.e. idempotent) elements we refer to [4]. We denote by $B(A)$ the set of all idempotents of A; $B(A)$ is the largest boolean subalgebra of A.

Since MV-algebras form a variety, the notions of MV-homomorphism, quotient, ideal are just the particular cases of the corresponding universal algebraic notions. For prime ideals, maximal ideals, and the radical ideal we refer to [4]. The set of all prime ideals of an MV-algebra A is denoted by $Spec(A)$, while $Max(A)$ and $Min(A)$ denote the sets of the maximal and minimal prime ideals of A, respectively. $Rad(A)$ denotes the radical ideal of A.

We can equip $Spec(A)$ with the Zariski topology. With such topology, $Spec(A)$ is a compact topological space [1]. With the induced topology, $Max(A)$ is a compact Hausdorff topological space.

For every $\emptyset \neq X \subseteq A$, we denote $X^{\perp} = \{y \in A \mid y \wedge x = 0, \text{ for every } x \in X\}$. Then $X^{\perp}$ is an ideal of A [1]. If $X = \{x\}$, then we write $x^{\perp}$ for $X^{\perp}$.

For local and simple MV-algebras we refer to [4].

An MV-algebra A is said to be *divisible* if and only if for any $x \in A$ and for any natural number $n \geqslant 1$ there is $y \in A$ such that $ny = x$ and $y \odot (n-1)y = 0$. Note that the second condition intuitively means $y \leqslant 1/n$, but not quite, since $1/n$ is not necessarily present in an MV-algebra. In every divisible MV-algebra, for each given x there is a unique y as above, and y will be denoted by $\delta_n(x)$.

It is worth to stress that divisible MV-algebras hold an important role in the proof of Chang's Completeness Theorem for MV-algebras. Moreover, the class of divisible MV-algebras is closed under quotient.

An ideal H of an MV-algebra A is called *primary* if it is contained in a unique maximal ideal. Note that for every ideal H, the quotient MV-algebra $\frac{A}{H}$ is local if and only if H is primary.

For every $P \in Spec(A)$, we define $O(P) = \bigcap\{m \in Min(A) \mid m \subseteq P\}$. It follows that $O(P)$ is an ideal of A such that $O(P) \subseteq P$. Moreover in [9] the following characterization of $O(P)$ is given: $O(P) = \bigcup\{a^{\perp} \mid a \notin P\}$. Moreover it results that $O(P)$ is a primary ideal for each $P \in Spec(A)$ and the intersection $\bigcap\{O(M) \mid M \in Max(A)\} = \{0\}$.

An MV-algebra A is said to be *regular* if and only if each minimal prime ideal is Stonean in the lattice reduct of A, i.e. it is generated by idempotent elements. In [12], Lacava proved that an MV-algebra is regular if and only if it is quasi-completely boolean dominated. Recall that an MV-algebra A is said to be *quasi-completely boolean dominated* if and only if, for each $x, y \in A$ such that $x \wedge y = 0$, there exist $b_1, b_2 \in B(A)$ such that $b_1 \geqslant x$, $b_2 \geqslant y$ and $b_1 \wedge b_2 = 0$. Moreover, the maximal spectrum of a regular MV-algebra is a Stone space with the Zariski topology [2].

Proposition 1. *Let A be a regular MV-algebra. For each $M \in Max(A)$, $O(M)$ is a minimal prime ideal.*

Proof. Remember that $O(M) = \bigcap\{m \in Min(A) \mid m \subseteq M\}$, for each $M \in Max(A)$. From Proposition 27 in [2], there exists a unique minimal prime ideal $m \subseteq M$, and so $O(M) = m$. $\square$