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Slow consistency

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ABSTRACT

The fact that "natural" theories, i.e. theories which have something like an "idea" to them, are almost always linearly ordered with regard to logical strength has been called one of the great mysteries of the foundation of mathematics. However, one easily establishes the existence of theories with incomparable logical strengths using self-reference (Rosser-style). As a result, **PA** + Con(**PA**) is not the least theory whose strength is greater than that of **PA**. But still we can ask: is there a sense in which **PA** + Con(**PA**) is the least "natural" theory whose strength is greater than that of **PA**? In this paper we exhibit natural theories in strength strictly between **PA** and **PA** + Con(**PA**) by introducing a notion of slow consistency.

1. Preliminaries

PA is Peano Arithmetic. **PA** $\lceil_k$ denotes the subtheory of **PA** usually denoted by $\mathrm{I}\Sigma_k$. It consists of a finite base theory $\mathbf{P}^-$ (which are the axioms for a commutative discretely ordered semiring) together with a single Π_{k+2} axiom which asserts that induction holds for Σ_k formulae. For functions $F:\mathbb{N}\to\mathbb{N}$ we use exponential notation $F^0(x)=x$ and $F^{k+1}(x)=F(F^k(x))$ to denote repeated compositions of F.

In what follows we require an ordinal representation system for ε_0. Moreover, we assume that these ordinals come equipped with specific fundamental sequences $\lambda[n]$ for each limit ordinal $\lambda\leqslant\varepsilon_0$. Their definition springs forth from their representation in Cantor normal form (to base ω). For an ordinal α such that $\alpha>0$, α has a unique representation:

$$\alpha=\omega^{\alpha_1}\cdot n_1+\cdots+\omega^{\alpha_k}\cdot n_k,$$

where $0<k,n_1,\ldots,n_k<\omega$, and $\alpha_1,\ldots,\alpha_k$ are ordinals such that $\alpha_1>\cdots>\alpha_k$.

If the Cantor normal form of $\beta>0$ is $\omega^{\beta_1}\cdot m_1+\cdots+\omega^{\beta_l}\cdot m_l$, we write $\alpha\gg\beta$ if $\alpha>\beta$ and $\alpha_k\geqslant\beta_1$.

Definition 1.1. For α an ordinal and n a natural number, let ω_n^α be defined inductively by $\omega_0^\alpha:=\alpha$, and $\omega_{n+1}^\alpha:=\omega^{\omega_n^\alpha}$.

We also write ω_n for ω_n^1. In particular, $\omega_0=1$ and $\omega_1=\omega$.

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Definition 1.2. For each limit ordinal $\lambda \leqslant \varepsilon_0$, define a strictly monotone sequence, $\lambda[n]$, of ordinals converging to λ from below. We use the fact, following from the Cantor normal form representation, that if $0 < \alpha < \varepsilon_0$, then there are unique $\beta, \gamma < \varepsilon_0$, and $0 < m < \omega$ such that

$$\alpha = \beta + \omega^{\gamma} \cdot m$$

and either $\beta = 0$ or β has normal form $\omega^{\beta_1} \cdot m_1 + \cdots + \omega^{\beta_l} \cdot m_l$ with $\beta_l > \gamma$.

The definition of $\lambda[n]$ proceeds by recursion on this representation of λ.

Case 1. $\lambda = \beta + \omega^{\gamma} \cdot m$ and $\gamma = \delta + 1$.

Put $\lambda[n] = \beta + \omega^{\gamma} \cdot (m-1) + \omega^{\delta} \cdot (n+1)$. (Remark: In particular, $\omega[n] = n+1$.)

Case 2. $\lambda = \beta + \omega^{\gamma} \cdot m$, and $\gamma < \lambda$ is a limit ordinal.

Put $\lambda[n] = \beta + \omega^{\gamma} \cdot (m-1) + \omega^{\gamma[n]}$.

Case 3. $\lambda = \varepsilon_0$.

Put $\varepsilon_0[0] = \omega$ and $\varepsilon_0[n+1] = \omega^{\varepsilon_0[n]}$. (Remark: Thus $\varepsilon_0[n] = \omega_{n+1}$.)

It will be convenient to have $\alpha[n]$ defined for non-limit α. We set $(\beta+1)[n] = \beta$ and $0[n] = 0$.

Definition 1.3. By "a fast growing" hierarchy we simply mean a transfinitely extended version of the Grzegorczyk hierarchy i.e. a transfinite sequence of number-theoretic functions $F_\alpha : \mathbb{N} \to \mathbb{N}$ defined recursively by iteration at successor levels and diagonalization over fundamental sequences at limit levels. We use the following hierarchy:

$$F_0(n) = n+1, \qquad F_{\alpha+1}(n) = F_{\alpha}^{n+1}(n), \qquad F_{\alpha}(n) = F_{\alpha[n]}(n) \quad \text{if } \alpha \text{ is a limit.}$$

It is closely related to the Hardy hierarchy:

$$H_0(n) = n, \qquad H_{\alpha+1}(n) = H_{\alpha}(n+1), \qquad H_{\alpha}(n) = H_{\alpha[n]}(n) \quad \text{if } \alpha \text{ is a limit.}$$

Their relationship is as follows:

$$H_{\omega^\alpha} = F_\alpha \tag{1}$$

for every $\alpha < \varepsilon_0$. If $\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k$ is in Cantor normal form and $\beta < \omega^{\alpha_k + 1}$, then

$$H_{\alpha+\beta} = H_\alpha \circ H_\beta. \tag{2}$$

Ketonen and Solovay [8] found an interesting combinatorial characterization of the H_α's. Call an interval $[k, n]$ 0-large if $k \leqslant n$, $\alpha + 1$-large if there are $m, m' \in [k, n]$ such that $m \neq m'$ and $[m, n]$ and $[m', n]$ are both α-large; and λ-large (where λ is a limit) if $[k, n]$ is $\lambda[k]$-large.

Theorem 1.4. *(See Ketonen, Solovay [8].) Let $\alpha < \varepsilon_0$.*

$$H_\alpha(n) = \textit{least } m \textit{ such that } [n, m] \textit{ is } \alpha\textit{-large}, \qquad F_\alpha(n) = \textit{least } m \textit{ such that } [n, m] \textit{ is } \omega^\alpha\textit{-large}.$$

The order of growth of F_{ε_0} is essentially the same as that of the Paris–Harrington function f_{PH}. More details will be provided in Section 3.1.

2. Capturing the F_α's in PA

In [8] many facts about the functions F_α, as befits their definition, are proved by transfinite induction on the ordinals $\leqslant \varepsilon_0$. In [8] there is no attempt to determine whether they are provable in **PA** (let alone in weaker theories). In what follows we will have to assume that some of the properties of the F_α's hold in all models of **PA**. As a consequence, we will revisit some parts of [8], especially Section 2, and recast them in such a way that they become provable in **PA**. Statements shown by transfinite induction on the ordinals in [8] will be proved by ordinary induction on the term complexity of ordinal representations, adding extra assumptions.

Definition 2.1. The computation of $F_\alpha(x)$ is closely connected with the step-down relations of [8] and [19]. For $\alpha < \beta \leqslant \varepsilon_0$ we write $\beta \underset{n}{\rightarrow} \alpha$ if for some sequence of ordinals $\gamma_0, \ldots, \gamma_r$ we have $\gamma_0 = \beta$, $\gamma_{i+1} = \gamma_i[n]$, for $0 \leqslant i < r$, and $\gamma_r = \alpha$. If we also want to record the number of steps r, we shall write $\alpha \overset{r}{\underset{n}{\rightarrow}} \beta$.

The definition of the functions F_α for $\alpha \leqslant \varepsilon_0$ employs transfinite recursion on α. It is therefore not immediately clear how we can speak about these functions in arithmetic. Later on we shall need to refer to a definition of $F_\alpha(x) = y$ in an arbitrary model of **PA**. As it turns out, this can be done via a formula of low complexity.