Annals of Pure and Applied Logic 164 (2013) 511–522

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Highness, locally noncappability and nonboundings ☆

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ARTICLE INFO

Article history:
Received 23 December 2011
Received in revised form 31 October 2012
Available online 4 December 2012

MSC:
03D10
03D20
03D25
03D28

Keywords:
Recursively enumerable degrees
Locally noncappable degrees
Nonboundings

ABSTRACT

In this paper, we improve a result of Seetapun and prove that above any nonzero, incomplete recursively enumerable (r.e.) degree **a**, there is a high$_2$ r.e. degree $\mathbf{c} > \mathbf{a}$ witnessing that **a** is locally noncappable (Theorem 1.1). Theorem 1.1 provides a scheme of obtaining high$_2$ nonboundings (Theorem 1.6), as all known high$_2$ nonboundings, such as high$_2$ degrees bounding no minimal pairs, high$_2$ plus-cuppings, etc.

1. Introduction

A recursively enumerable (r.e.) degree **a** is *locally noncappable* if there is an r.e. degree **c** above **a** such that no nonzero r.e. degree **w** below **c** forms a minimal pair with **a**. We say that **c** witnesses that **a** is locally noncappable.

Seetapun proved in his thesis [10] that every nonzero incomplete r.e. degree is locally noncappable. Giorgi published Seetapun's result in [5], but with one Σ_3 outcome missing, therefore, we can say that Giorgi's construction is not complete. In this paper, we improve Seetapun's result by showing that such witnesses can always be chosen as high$_2$ degrees.

Theorem 1.1. *Given a nonzero incomplete r.e. degree* **a**, *there exists a high$_2$ r.e. degree* $\mathbf{c} > \mathbf{a}$ *witnessing that* **a** *is locally noncappable.*

The proof of Theorem 1.1 combines Seetapun's construction and the high$_2$ strategy developed in Lerman [8] and Downey, Lempp and Shore [2]. As expected, our construction contains new features in the gap–cogap argument, where after a gap is closed unsuccessfully, it can be opened again, due to the changes of A. Theorem 1.1 is strong enough, it can have all the known high$_2$ nonboundings as corollaries.

Corollary 1.2. *(See Downey, Lempp and Shore [2].) There is a high$_2$ r.e. degree bounding no minimal pairs.*

☆ Stephan is partially supported by NUS grants R252-000-308-112 and R252-000-420-112. Wu is partially supported by AcRF grants RG37/09, M52110101 and MOE2011-T2-1-071 (ARC 17/11, M45110030) from Ministry of Education of Singapore. The authors would like to thank the anonymous referee for many good suggestions for improving the paper.

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0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2012.11.008

Let **a** be a nonbounding degree (see Soare [12]), and apply Theorem 1.1 to **a** and the obtained **c** is a high_2 nonbounding degree.

In contrast to noncuppable degrees, Harrington proposed the notion of plus-cupping degrees, where a nonzero r.e. degree **a** is plus-cupping if for every nonzero r.e. degree **b** below **a** and for every r.e. degree **c** above **a**, there is an r.e. degree **d** below **c** such that $\mathbf{b} \cup \mathbf{d} = \mathbf{c}$ (Soare [12, p. 387]). By applying a $\mathbf{0}'''$ argument, Harrington was able to show the existence of plus-cupping degrees. In [4], Fejer and Soare rephrased the concept of plus-cupping by restricting **c** to $\mathbf{0}'$, which is weak version of Harrington's plus-cupping. It turns out that the construction of Fejer–Soare's plus-cupping degree is easier, and is now a standard illustration of gap–cogap arguments. We apply Theorem 1.1 to Harrington plus-cupping degrees. Note that the existence of a high_2 Harrington plus-cupping degrees was used by Li to show that the class of high r.e. degrees and the class of high_n-degrees ($n \geqslant 2$) are not elementarily equivalent.

Corollary 1.3. *Above any Harrington plus-cupping degree, there is a* high_2 *Harrington plus-cupping degree.*

We can also treat Harrington plus-cupping degrees and Fejer–Soare's plus-cupping degrees as nonbounding phenomenon. For instance, if **a** is a Fejer–Soare's plus-cupping degree, then it bounds no nonzero noncuppable degrees.

In [7], Leonhardi considered Slaman triples and proposed a new nonbounding. Here we say that r.e. degrees **a**, **b**, **c** form a Slaman triple, if $\mathbf{a} > \mathbf{0}$, $\mathbf{c} \not\geqslant \mathbf{b}$ and for any $\mathbf{w} \leqslant \mathbf{a}$, if $\mathbf{w} > \mathbf{0}$, then $\mathbf{w} \cup \mathbf{c} \geqslant \mathbf{b}$. **a** is called the base of this Slaman triple.

Corollary 1.4. *(See Leonhardi [7].) There is a* high_2 *degree bounding no bases of Slaman triples.*

Again, if we have a nonzero r.e. degree **a** bounding no bases of Slaman triples, then by applying Theorem 1.1 to **a**, we can get a high_2 one.

Furthermore, Theorem 1.1 can be applied to obtain some new results: no nonbounding degrees mentioned above can be maximal.

Theorem 1.5. *There are no maximal nonbounding degrees for minimal pairs* (*first proved by Seetapun*)*, plus-cupping degrees* (*both versions*)*, and bases for Slaman triples, respectively.*

Theorem 1.1 can be used as a scheme for constructing high_2 nonbounding degrees. That is, in general, if we have a property P about the r.e. degrees, we say that an r.e. degree **a** has plus-P property, if for any nonzero r.e. degree $\mathbf{b} \leqslant \mathbf{a}$, $P(\mathbf{b})$ is also true.

Theorem 1.6. *Suppose that* P *is a closed-upwards property in the r.e. degrees. If* **a** *has plus-*P *property, then above* **a**, *there is a* high_2 *degree* **c** *with plus-*P *property.*

The proof of Theorem 1.6 is quite easy. By Theorem 1.1, there exists a high_2 degree $\mathbf{c} > \mathbf{a}$ witnessing the **a** is noncappable below **c**.

We claim that **c** has plus-P property. Suppose not. Then there is a nonzero r.e. degree $\mathbf{b} < \mathbf{c}$ which does not have property P. Then **a** and **b** form a minimal pair, as otherwise, there exists a nonzero r.e. degree **w** below both **a** and **b**, and hence, due to the assumption that **a** has plus-P property, **w** has P property, which implies that **b** has P property. A contradiction.

However, by the choice of **c**, **a** is noncappable below **c**, we get another contradiction. This shows that **c** has plus-P property.

Besides this, by applying almost the same argument, we can apply Theorem 1.1 to prove the continuity of capping property of r.e. degrees, which was first proved by Harrington and Soare in [6].

Corollary 1.7. *For any r.e. degrees* **a** *and* **b**, *if* **a** *and* **b** *form a minimal pair, then there exists a* high_2 *degree* **c** *above* **b** *such that* **c** *and* **a** *form a minimal pair.*

The continuity property of bases of Slaman triples is also true.

Corollary 1.8. *If* $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *form a Slaman triple, then there is a* high_2 *r.e. degree* **e** *above* **a** *such that* $(\mathbf{e}, \mathbf{b}, \mathbf{c})$ *also form a Slaman triple.*

Note that Theorem 1.1 is optimal, as there is no way to make the witness **c** high_2. To see this, let **a** be a nonzero r.e. degree bounding no minimal pairs, then **c** also bounds no minimal pairs, which implies that **c** cannot be high, as each high r.e. degree bounds a minimal pair, by Cooper [1] and Shore and Slaman [11].

We notice that Downey and Stob proved in 1997 in [3] that any nonzero incomplete c.e. degree **c** is a witness of local noncappability of a c.e. degree **a** below **c**.

Our notation and terminology are standard and generally follow Odifreddi [9] and Soare [12]. We say that a number is big in the construction if it is the least natural number (in an effective way) greater than any numbers mentioned so far.