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Computability of the ergodic decomposition

Mathieu Hoyrup

LORIA, INRIA Nancy-Grand Est, 615 rue du jardin botanique, 54600 Villers-lès-Nancy, France

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ABSTRACT

The study of ergodic theorems from the viewpoint of computable analysis is a rich field of investigation. Interactions between algorithmic randomness, computability theory and ergodic theory have recently been examined by several authors. It has been observed that ergodic measures have better computability properties than non-ergodic ones. In a previous paper we studied the extent to which non-ergodic measures inherit the computability properties of ergodic ones, and introduced the notion of an *effectively decomposable* measure. We asked the following question: if the ergodic decomposition of a stationary measure is finite, is this decomposition effective? In this paper we answer the question in the negative.

1. Introduction

The ergodic decomposition theorem says the following: every stationary process can be decomposed into ergodic processes, such that almost every realization of the original process can be seen as a realization of one of the ergodic processes, chosen at random. Ergodic processes are in a sense the building blocks of all the stationary processes. The question of the effectiveness of many ergodic theorems has received much attention in recent years and it progressively appeared that ergodic measures behave differently from non-ergodic ones. For instance, the speed of convergence of Birkhoff averages is computable in the ergodic case [1] while it is not computable in general [16]; Birkhoff ergodic theorem holds exactly at Schnorr random sequences in the ergodic case [8] and at Martin-Löf random sequences in general [16,4]. These examples suggest that ergodic measures have better computability properties than non-ergodic ones. In [10] we showed that the sticking point is not really ergodicity but the computability of the ergodic decomposition. While every non-ergodic measure has a unique decomposition into ergodic ones, this decomposition is not always computable. The known examples of non-ergodic measures whose decomposition is non-computable are infinite combinations of ergodic measures [16,1]. In [10] we raised the following question: if the decomposition of a non-ergodic measure is finite, is this decomposition computable? In the present paper we solve the problem and show that it is not necessarily true. Before presenting this new result, we review the results obtained in [10] and characterize the effective compact classes of ergodic measures.

The paper is organized as follows. In Section 2 we give the necessary background on computability and randomness. In Section 3 we develop results about randomness and combinations of measures that will be applied in the sequel, but are of independent interest (i.e., outside ergodic theory). We start Section 4 with a reminder on the ergodic decomposition and then relate it to randomness. In Section 5 we study the particular case of effective compact classes of ergodic measures. We finish in Section 6 by our main result: there exist ergodic measures P and Q whose average is not effectively decomposable.

E-mail address: mathieu.hoyrup@inria.fr.

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2. Preliminaries

We assume familiarity with algorithmic randomness and computability theory. For more details on computable analysis we refer the reader to [17].

2.1. Computability

A computable metric space is a triple (X, d, S) where (X, d) is a complete separable metric space and S is a countable dense subset together with a fixed numbering such that for all $s, s' \in S$, $d(s, s')$ is a computable real number, uniformly in the indices of s and s'. The basic metric balls $B(s, q)$ with $s \in S$ and $q \in \mathbb{Q}_{>0}$ form a countable basis of the topology induced by the metric d. We fix a canonical effective numbering $(B_i)_{i\in\mathbb{N}}$ of this basis.

Let X be a computable metric space. A *name* for $x \in X$ is a sequence $s_n \in S$ such that $d(s_n, x) < 2^{-n}$. A point x is *computable* if it has a computable name. A set $U \subseteq X$ is an *effective open set* if there is r.e. set $E \subseteq \mathbb{N}$ such that $U = \bigcup_{i\in E} B_i$. A function $f : X \to Y$ is computable if there is a machine that, provided a name for x as oracle, computes a name for $f(x)$. Equivalently, f is computable if the pre-images $f^{-1}(B_i)$ are effective open sets, uniformly in i. Let $A \subseteq X$. A function $f : A \to Y$ is *computable on A* if there is a machine that, provided a name for $x \in A$ as oracle, computes a name for $f(x)$. Equivalently, f is computable on A if the pre-images $f^{-1}(B_i) \cap A$ are intersections of uniformly effective open sets with A. A point $y \in Y$ is *computable relative to* $x \in X$ if the function $x \mapsto y$ is computable on $\{x\}$. A function $f : X \to [0, +\infty]$ is *lower semi-computable* if there is a machine that, provided a name for x as oracle, computes a nondecreasing sequence of rational numbers converging to $f(x)$. Equivalently f is lower semi-computable if the pre-images $f^{-1}(q, +\infty]$ are effective open sets, uniformly in $q \in \mathbb{Q}$. A compact set $K \subseteq X$ is *effectively compact* if the set of finite unions of balls covering K is r.e.

We will use the following simple results that are the effective counterparts of basic topological properties.

Fact 1 *(Folklore).*

1. The complement of an effective compact set is an effective open set.
2. If K is effectively compact and U effectively open then $K \setminus U$ is effectively compact.

Proof.

1. Let $K \subseteq X$ be effectively compact. Let B be a basic metric ball and $\overline{B}$ be the corresponding closed ball. As the complement of $\overline{B}$ is effectively open so $K \cap \overline{B} = \emptyset$ can be semi-decided. Hence $X \setminus K$ is the r.e. union of all basic balls B such that $K \cap \overline{B} = \emptyset$.
2. $K \setminus U$ is compact and $K \setminus U \subseteq (B_1 \cup \cdots \cup B_n) \iff K \subseteq U \cup (B_1 \cup \cdots \cup B_n)$ which can be semi-decided. □

Let $K \subseteq X$ be an effective compact set and $f : K \to Y$ a function computable on K.

Fact 2 *(Folklore).* $f(K)$ is an effective compact set.

Proof. Let $B_1, \ldots, B_n$ be basic balls of Y. $f(K)$ is contained in $B_1 \cup \cdots \cup B_n$ if and only if K is contained in $f^{-1}(B_1 \cup \cdots \cup B_n)$, which is an effective open set. As K is effectively compact the latter inclusion can be semi-decided. □

Fact 3 *(Folklore).* If f is moreover one-to-one then $f^{-1} : f(K) \to K$ is computable on $f(K)$.

Proof. For the sake of clarity, we denote f^{-1} by g.

Let $B \subseteq X$ be a basic ball. We have to prove that there is an effective open set $V \subseteq Y$ such that $g^{-1}(B) = V \cap f(K)$. The set $C := K \setminus B$ is an effective compact set. $g^{-1}(B) = g^{-1}(K \setminus C) = g^{-1}(K) \setminus g^{-1}(C) = f(K) \setminus f(C)$. As C is an effective compact set, its complement V is an effective open set and $g^{-1}(B) = f(K) \cap V$. As everything is uniform in B, g is computable. □

The product of two computable metric spaces has a natural structure of computable metric space.

Fact 4 *(Folklore).* If $K \subseteq X$ is an effective compact set and $f : K \times Y \to \overline{\mathbb{R}}$ is lower semi-computable, then the function $g : Y \to \overline{\mathbb{R}}$ defined by $g(y) = \inf_{x\in K} f(x, y)$ is lower semi-computable.

Proof. Let us prove that $g^{-1}(q, +\infty] = \{y\colon\ K \times \{y\} \subseteq f^{-1}(q, +\infty]\}$ is an effective open set, uniformly in q. Let q be some fixed rational number. The effective open set $U_q = f^{-1}(q, +\infty]$ can be expressed as an effective union of product balls $U_q = \bigcup_{i\in\mathbb{N}} B^X_i \times B^Y_i$. The set $E_q = \{(i_1, \ldots, i_k)\colon\ K \subseteq B^X_{i_1} \cup \cdots \cup B^X_{i_k}\}$ is r.e. and it is easy to prove that $g^{-1}(q, +\infty] = \bigcup_{(i_1,\ldots,i_k)\in E_q} B^Y_{i_1} \cap \cdots \cap B^Y_{i_k}$, which is an effective open set. The argument is uniform in q. □