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Integration in algebraically closed valued fields with sections

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ABSTRACT

We construct Hrushovski–Kazhdan style motivic integration in certain expansions of ACVF. Such an expansion is typically obtained by adding a full section or a cross-section from the RV-sort into the VF-sort and some (arbitrary) extra structure in the RV-sort. The construction of integration, that is, the inverse of the lifting map $\mathbb{L}$, is rather straightforward. What is a bit surprising is that the kernel of $\mathbb{L}$ is still generated by one element, exactly as in the case of integration in ACVF. The overall construction is more or less parallel to the main construction of Hrushovski and Kazhdan (2006) [10], as presented in Yin (2010) [19] and Yin (2011) [20]. As an application, we show uniform rationality of Igusa zeta functions for non-archimedean local fields with unbounded ramification degrees.

1. Introduction

We have presented the main construction of the Hrushovski–Kazhdan integration theory [10] in [19,20]. The integration constructed there is "unrefined" in the sense that, although the kernel of the lifting map $\mathbb{L}$, that is, the congruence relation I_{sp}, is surprisingly simple, being generated by a single element, and the whole theory is structurally sound, satisfying, among other things, a Fubini-type theorem and a change of variables formula, computation of most integrals appear to be too complicated or utterly intractable. This is so even without volume forms and when only simple geometrical objects are involved, such as an open ball with one closed hole and a closed ball with two open holes, computing the standard contractions of which, according to [19, Proposition 6.18], would tell us whether there is a definable bijection of the two in ACVF. Refinement may proceed in several directions, for example, see [10, §10] and [11], all of which involve manipulations of the Grothendieck (semi)rings that provide values for motivic integrals, such as groupifying, coarsening (usually by way of introducing external algebraic structures), and decomposing into tensor product. This last manipulation makes computation of certain integrals much more transparent, especially when integrating functions with one variable, such as the one mentioned above.

In this paper we shall first construct "unrefined" motivic integration maps in certain expansions of algebraically closed valued fields and then refine the target semirings of these maps by decomposing them into tensor products in a canonical way. Such an expansion of algebraically closed valued fields is typically obtained in two independent steps: adding a full section (an RV-section) or a cross-section from the RV-sort into the VF-sort and then adding arbitrary relations and functions in the RV-sort. Expansions with extra structure in the RV-sort has been considered in [10, §12], where a homomorphism

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between Grothendieck semirings is obtained more or less along the line of the main construction, in particular, the congruence relation I_{sp} retains the same degree of simplicity. Expansions with a section from the residue field into the valued field (a $\overline{\mathrm{K}}$-section) has been considered in [12]. This is in the context of adelic structures over curves, where an integration in the style of [10] is not needed and hence is not developed.

Our motivation for extending the Hrushovski–Kazhdan theory to such expansions is twofold. Firstly, this is to prepare the ground for a plausible theory of motivic characters, especially multiplicative ones, which is something we should have if we are to further the (already far-reaching) application of the theory of motivic integration to, say, geometry and representation theory, as demonstrated, for example, in [1–3,12,13]. The use of characters in constructing representations in function spaces is beautifully expounded in the (perhaps a bit old-fashioned but still tremendously insightful) work [9]. Secondly, motivic integration in real closed fields is alluded to in the introduction of [10] as a hope. We shall realize this hope in a future paper [21]. The framework for doing so calls for a cross-section and its technical aspects closely resemble those of this paper.

The construction in this paper is entirely modeled on and heavily relies on the (auxiliary) results of the construction presented in [19,20]. In particular, we still adhere to the three-step procedure as laid out in the introduction of [20]. For clarity, let us repeat it once again. Let $\mathbb{T}$ be an expansion of ACVF, which includes an RV-section $\mathrm{sn} : \mathrm{RV} \longrightarrow \mathrm{VF}$ or a cross-section $\mathrm{csn} : \Gamma \longrightarrow \mathrm{VF}$ or both. Let VF_* and $\mathrm{RV}[*]$ be two suitable categories of definable sets that are respectively associated with the VF-sort and the RV-sort. To construct a canonical homomorphism from the Grothendieck semigroup $\mathbf{K}_+ \mathrm{VF}_*$ to the Grothendieck semigroup $\mathbf{K}_+ \mathrm{RV}[*]/\mathrm{I}_{\mathrm{sp}}$, where I_{sp} is a suitable semigroup congruence relation, we proceed as follows:

- *Step* 1. There is a natural lifting map $\mathbb{L}$ from the set of objects of $\mathrm{RV}[*]$ into the set of objects of VF_*. We show that $\mathbb{L}$ hits every isomorphism class of VF_*.
- *Step* 2. We show that $\mathbb{L}$ induces a semigroup homomorphism from $\mathbf{K}_+ \mathrm{RV}[*]$ into $\mathbf{K}_+ \mathrm{VF}_*$, which is also denoted by $\mathbb{L}$.
- *Step* 3. In order to obtain a precise description of the semigroup congruence relation on $\mathbf{K}_+ \mathrm{RV}[*]$ induced by $\mathbb{L}$, that is, the kernel of $\mathbb{L}$, we introduce two operations: special bijection in the VF-sort and blowup in the RV-sort. In a sense these two operations mirror each other. Using this correlation we show that, for any objects $\mathbf{U}_1$, $\mathbf{U}_2$ in $\mathrm{RV}[*]$, there are isomorphic blowups $\mathbf{U}_1^\sharp$, $\mathbf{U}_2^\sharp$ of $\mathbf{U}_1$, $\mathbf{U}_2$ if and only if $\mathbb{L}(\mathbf{U}_1)$, $\mathbb{L}(\mathbf{U}_2)$ are isomorphic.

Through certain standard algebraic manipulations, the inverse of $\mathbb{L}$ gives rise to various ring homomorphisms and module homomorphisms. These are understood as generalized Euler characteristic or, if volume forms are present, integration. Note that, in principle, the construction is already completed in Step 2 (see Section 4). However, to facilitate computation in future applications, it seems much more satisfying to have a precise description of the semigroup congruence relation as obtained in Step 3 (see Section 5). Perhaps a bit surprisingly, this kernel of $\mathbb{L}$ is still generated by one element, exactly as in the case of integration in ACVF.

There is really just one new (nontechnical) idea in this paper, which is very straightforward. For every $\mathbb{T}$-definable set A we seek a definable function $\pi : A \longrightarrow \mathrm{RV}^m$ such that each fiber $\pi^{-1}(\vec{t})$ is $\mathrm{sn}(\vec{t})$-definable in ACVF, similarly if the RV-section sn is replaced by the cross-section csn (we have to work with csn instead of sn in the situation with volume forms). Such a function is called an RV- or a Γ-partition of A. If it exists then we may assign a volume to A by first computing the volumes of the fibers, using the results for ACVF, and then sum them up more or less formally. In fact such a partition always exists for a definable set. Conceptually, the few foregoing sentences capture the gist of this paper so well that it is actually tempting to end the discussion right here. But that is probably not very convincing for someone who is not already familiar with the intricate working of the Hrushovski–Kazhdan theory, especially when highly nontrivial modifications of certain technical results are called for. So, we opt for spelling out more details in a few pages. Inevitably, the writing will repeat (variations of) some things that have already been said in [19,20].

In [18] we have compared expansions with RV-section and expansions with $\overline{\mathrm{K}}$-section in terms of minimality conditions. It is not hard to see that our method here also works for expansions of ACVF with $\overline{\mathrm{K}}$-section.

We now describe an application to local zeta functions. Let $f(\vec{X}) \in \mathbb{Q}_p[X_1, \ldots, X_n]$, κ be a positive real number, and L be a finite extension of $\mathbb{Q}_p$. The norm of $a \in L$ is denoted by $|a|_L$ and the Haar measure on L is denoted by $|\mathrm{d}\vec{X}|_L$. Suppose that $A \subseteq L^n$ is bounded and is $\mathbb{Q}_p$-definable in the language with a cross-section. Note that here the parameters used to define f and A are allowed to vary in a suitable way as p and L vary, for example, the ramification degree of L may be a defining parameter for A. Consider the Igusa local zeta function

$$\zeta(A, L, \kappa) = \int_A \left|f(\vec{X})\right|_L^\kappa |\mathrm{d}\vec{X}|_L.$$

Following the specialization procedure in [10], we can show that $\zeta(A, L, \kappa)$ is uniformly rational for all p-adic fields (see Definition 6.4 for the precise meaning of uniformity). This can also be derived using the Denef–Pas method in [6,7,16,17].

The paper is organized as follows. In Section 2 we first introduce the class of expansions of ACVF that shall be considered. Obviously not much can be done without quantifier elimination, which is derived immediately. Other basic structural properties are also collected in this section, which shall be used throughout the rest of the paper. In Section 3 categories associated with the RV-sort are introduced and their Grothendieck semigroups are studied. Here the reader should notice that, by having a cross-section, the various target semirings of the Grothendieck homomorphisms actually become simpler