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The isomorphism problem for ω-automatic trees

Dietrich Kuske [a], Jiamou Liu [b], Markus Lohrey [c,*]

[a] TU Ilmenau, Institut für Theoretische Informatik, Germany
[b] Auckland University of Technology, New Zealand
[c] Universität Leipzig, Institut für Informatik, Germany

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ABSTRACT

The main result of this paper states that the isomorphism problem for ω-automatic trees of finite height is at least has hard as second-order arithmetic and therefore not analytical. This strengthens a recent result by Hjorth, Khoussainov, Montalbán, and Nies (2008) [12] showing that the isomorphism problem for ω-automatic structures is not in Σ^1_2. Moreover, assuming the continuum hypothesis **CH**, we can show that the isomorphism problem for ω-automatic trees of finite height is recursively equivalent with second-order arithmetic. On the way to our main results, we show lower and upper bounds for the isomorphism problem for ω-automatic trees of every finite height: (i) It is decidable (Π^0_1-complete, resp.), for height 1 (2, resp.), (ii) Π^1_1-hard and in Π^1_2 for height 3, and (iii) Π^1_{n-3}- and Σ^1_{n-3}-hard and in Π^1_{2n-4} (assuming **CH**) for height $n \geqslant 4$. All proofs are elementary and do not rely on theorems from set theory.

1. Introduction

A graph is computable if its domain is a computable set of natural numbers and the edge relation is computable as well. Hence, one can compute effectively in the graph. On the other hand, practically all other properties are undecidable for computable graphs (e.g., reachability, connectedness, and even the existence of isolated nodes). In particular, the isomorphism problem is highly undecidable in the sense that it is complete for Σ^1_1 (the first existential level of the analytical hierarchy [25]); see e.g. [5,10] for further investigations of the isomorphism problem for computable structures. These algorithmic deficiencies have motivated in computer science the study of more restricted classes of finitely presented infinite graphs. For instance, pushdown graphs, equational graphs, and prefix recognizable graphs have a decidable monadic second-order theory and for the former two the isomorphism problem is known to be decidable [7] (for prefix recognizable graphs the status of the isomorphism problem seems to be open).

Automatic graphs [16] are in between prefix recognizable and computable graphs. In essence, a graph is automatic if the elements of the universe can be represented as strings from a regular language and the edge relation can be recognized by a finite state automaton with several heads that proceed synchronously. Automatic graphs (and more general, automatic structures) received increasing interest over the last years [3,13,17,18,29,1]. One of the main motivations for investigating automatic graphs is that their first-order theories can be decided uniformly (i.e., the input is an automatic presentation and a first-order sentence). On the other hand, the isomorphism problem for automatic graphs is Σ^1_1-complete [17] and

* Corresponding author.
E-mail addresses: dietrich.kuske@tu-ilmenau.de (D. Kuske), jiamou.liu@aut.ac.nz (J. Liu), lohrey@informatik.uni-leipzig.de (M. Lohrey).

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hence as complex as for computable graphs (see [23] for the recursion theoretic complexity of other natural properties of automatic graphs).

In our recent paper [21], we studied the isomorphism problem for restricted classes of automatic graphs. Among other results, we proved that: (i) the isomorphism problem for automatic trees of height at most $n \geqslant 2$ is complete for the level Π^0_{2n-3} of the arithmetical hierarchy, (ii) that the isomorphism problem for well-founded automatic order trees is recursively equivalent to true arithmetic, and (iii) that the isomorphism problem for automatic order trees is Σ^1_1-complete. In this paper, we extend our techniques from [21] to *ω-automatic trees*. The class of ω-automatic structures was introduced in [2]; it generalizes automatic structures by replacing ordinary finite automata by Büchi-automata on ω-words. In this way, uncountable graphs can be specified. Some recent results on ω-automatic structures can be found in [22,12,14,19]. On the logical side, many of the positive results for automatic structures carry over to ω-automatic structures [2,14]. On the other hand, the isomorphism problem of ω-automatic structures is more complicated than that of automatic structures (which is Σ^1_1-complete). Hjorth et al. [12] constructed two ω-automatic structures for which the existence of an isomorphism depends on the axioms of set theory. Using Schoenfield's absoluteness theorem, they infer that isomorphism of ω-automatic structures does not belong to Σ^1_2. The extension of our elementary techniques from [21] to ω-automatic trees allows us to show directly (without a "detour" through set theory) that the isomorphism problem for ω-automatic trees of finite height is not analytical (i.e., does not belong to any of the levels Σ^1_n). For this, we prove that the isomorphism problem for ω-automatic trees of height $n \geqslant 4$ is hard for both levels Σ^1_{n-3} and Π^1_{n-3} of the analytical hierarchy (our proof is uniform in n). A more precise analysis moreover reveals at which height the complexity jump for ω-automatic trees occurs: For automatic as well as for ω-automatic trees of height 2, the isomorphism problem is Π^0_1-complete and hence arithmetical. But the isomorphism problem for ω-automatic trees of height 3 is hard for Π^1_1 (and therefore outside of the arithmetical hierarchy) while the isomorphism problem for automatic trees of height 3 is Π^0_3-complete [21]. Our lower bounds for ω-automatic trees even hold for the restricted class of injectively ω-automatic trees.

We prove our results by reductions from monadic second-order (fragments of) number theory. The first step in the proof is a normal form for analytical predicates. The basic idea of the reduction then is that a subset $X \subseteq \mathbb{N}$ can be encoded by an ω-word w_X over $\{0,1\}$, where the i-th symbol is 1 if and only if $i \in X$. The combination of this basic observation with our techniques from [21] allows us to encode monadic second-order formulas over $(\mathbb{N},+,\times)$ by ω-automatic trees of finite height. This yields the lower bounds mentioned above. We also give an upper bound for the isomorphism problem: for ω-automatic trees of height n, the isomorphism problem belongs to Π^1_{2n-4}. While the lower bound holds in the usual system **ZFC** of set theory, we can prove the upper bound only assuming in addition the continuum hypothesis. The precise recursion theoretic complexity of the isomorphism problem for ω-automatic trees remains open, it might depend on the underlying axioms for set theory.

Related work. Results on isomorphism problems for various subclasses of automatic structures can be found in [17,18,21,28]. Some completeness results for low levels of the analytical hierarchy for decision problems on infinitary rational relations were shown in [8]. In [9], it was shown that the isomorphism problems for ω-tree-automatic boolean algebras, (commutative) rings, and nilpotent groups of class $n > 1$ neither belong to Σ^1_2 nor to Π^1_2.

2. Preliminaries

Let $\mathbb{N}_+ = \{1,2,3,\ldots\}$ be the set of naturals without 0. With $\bar{x}$ we denote a tuple $(x_1,\ldots,x_m)$ of variables, whose length m does not matter.

2.1. The analytical hierarchy

In this paper we follow the definitions of the arithmetical and analytical hierarchy from [25]. In order to avoid some technical complications, it is useful to exclude 0 in the following, i.e., to consider subsets of $\mathbb{N}_+$. In the following, f_i ranges over unary functions on $\mathbb{N}_+$, X_i over subsets of $\mathbb{N}_+$, and $u,x,y,z,x_i,\ldots$ over elements of $\mathbb{N}_+$. The class $\Sigma^0_n \subseteq 2^{\mathbb{N}_+}$ is the collection of all sets $A \subseteq \mathbb{N}_+$ of the form

$$A = \bigl\{x \in \mathbb{N}_+ \bigm| (\mathbb{N},+,\times) \models \exists y_1 \forall y_2 \cdots Q\, y^1_n, y^2_n, \ldots, y^m_n\colon \varphi\bigl(x, y_1, \ldots, y^1_n, y^2_n, \ldots, y^m_n\bigr)\bigr\},$$

where $Q = \forall$ (resp. $Q = \exists$) if n is even (resp. odd) and φ is a quantifier-free formula over the signature containing $+$ and $\times$. The class Π^0_n is the class of all complements of Σ^0_n sets. The classes Σ^0_n, Π^0_n $(n \geqslant 1)$ make up the *arithmetical hierarchy*.

The analytical hierarchy extends the arithmetical hierarchy and is defined analogously using function quantifiers: The class $\Sigma^1_n \subseteq 2^{\mathbb{N}_+}$ is the collection of all sets $A \subseteq \mathbb{N}_+$ of the form

$$A = \bigl\{x \in \mathbb{N}_+ \bigm| (\mathbb{N},+,\times) \models \exists f_1 \forall f_2 \cdots Q\, f_n\colon \varphi(x, f_1, \ldots, f_n)\bigr\}, \tag{1}$$

where $Q = \forall$ (resp. $Q = \exists$) if n is even (resp. odd) and φ is a first-order formula over the signature containing $+$, $\times$, and the functions $f_1,\ldots,f_n$. The class Π^1_n is the class of all complements of Σ^1_n sets. The classes Σ^1_n, Π^1_n $(n \geqslant 1)$ make up