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Forcing $\square_{\omega_1}$ with finite conditions [☆]

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ABSTRACT

We give a construction of the square principle $\square_{\omega_1}$ by means of forcing with finite conditions.

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1. Introduction

The square principle on a cardinal κ states that there is a sequence $\langle C_\alpha \rangle_\alpha$ indexed by the limit ordinals in $[\kappa, \kappa^+)$ such that each C_α is a club subset of α of order type $\leqslant \kappa$ and the sequence is coherent in the sense that if β is a limit point of α then $C_\beta = C_\alpha \cap \beta$. This principle is a feature of the constructible universe **L** which was discovered by Jensen and used by him to show the existence of an ω_2-Souslin tree in **L** [7]. The related principle $\diamondsuit$, which was used to construct an ω_1-Souslin tree in **L** by Jensen, may be added or destroyed by forcing as wished (see [10] for examples and discussion). Also, by recent work of Shelah [12], at $\kappa \geqslant \omega_2$ which are successor cardinals of the form $\kappa = \theta^+ = 2^\theta$, $\diamondsuit_\kappa$ simply holds, i.e. it is equivalent to the cardinal arithmetic assumption $\theta^+ = 2^\theta$. However, $\square$ is connected to large cardinals. For example, by a well-known result of Solovay et al. [13], square cannot hold above a supercompact cardinal, and on smaller cardinals, it cannot hold in the presence of forcing axioms, e.g. Todorčević [14] proved that PFA implies that for all $\kappa \geqslant \omega_2$, $\square_\kappa$ fails. Therefore $\square$ can be seen as a reflection principle inimical to large cardinals, and in fact by varying the definition of square by allowing a cardinal parameter which measures how many guesses to C_α we are allowed at each α, we obtain a hierarchy

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of principles of decreasing strength which can be used to test consistency strength of various principles (see more on this in [3]). In the light of these facts it is natural that the question of how to add or destroy a square principle by forcing has been a central theme. See [3] for a description of some of the many known results including versions of an older result of Jensen and Magidor in which a square sequence is added by forcing.

One way to add a square, due to Jensen, is to force by initial segments along a closed unbounded subset of the domain, and to use the existence of the "top" point in the domain of a forcing condition to show that the forcing is strategically closed. Note that the principle $\Box_\omega$ is trivially true, by taking C_α to be any club of α of order type ω, so the first nontrivial instance of square is $\Box_{\omega_1}$. The method of forcing by initial segments means that to get $\Box_{\omega_1}$ we need to force with conditions whose domain has size ω_1. The referee has kindly informed us that in an unpublished work Foreman and Magidor added square by a countably closed forcing using countable conditions. A condition p in their forcing prescribes C_α for α of countable cofinality in $\mathrm{dom}(p)$, and for $\alpha \in \mathrm{dom}(p)$ of uncountable cofinality, p prescribes an initial segment of C_α which goes past $\sup(\mathrm{dom}(p) \cap \alpha)$. Assuming CH this poset has the ω_2-c.c. In this work we have been interested in another way of adding a square, using conditions whose domain is a finite set. The interest in doing this stems from a need to understand how one can control a one cardinal gap in forcing notions, which is a subject that has been of interest for various combinatorial issues for a long time. A glaring example of the need to develop this subject is the combinatorics of the structure $(\omega_1^{\omega_1}, \leqslant_{\mathrm{Fin}})$, which in contrast with the vast body of knowledge about $(\omega^\omega, \leqslant_{\mathrm{Fin}})$, remains a mysterious object. An important development on the subject of $(\omega_1^{\omega_1}, \leqslant_{\mathrm{Fin}})$ is Koszmider's paper [9] in which he shows that it is consistent to have an increasing chain of length ω_2 in this structure. Koszmider's paper also gives an overview of the difficulties that there are in forcing one gap results.

Koszmider's method is to force with conditions where a morass is used as a side condition. Our method is more directly connected to a different approach, which was used to force a club on ω_2 using finite conditions. This was done in two different but similar ways by Friedman in [5] and Mitchell in [11]. Both approaches are built upon a version of adding a club subset of ω_1 using finite conditions, as discovered by Baumgartner et al. [2] and modified by Abraham and Shelah in [1]. The main idea in Baumgartner et al.'s approach is that to force a club in ω_1 and avoid problems at the limit stages, one needs to specify by each condition not only what will go in the club, but also whole intervals that need to stay out of it. At ω_2 one can do the same, but now one needs to add side conditions in the form of coherent systems of models in order to make sure that cardinals are preserved, as was first done by Todorčević in [15]. This already is technically rather involved. What we have done is add to this the coherent partial square sequence. Namely, we actually force a square indexed by a club set. The existence of such a square implies the existence of an actual square sequence. This club set is like the one added by Friedman and Mitchell. The actual forcing notion needs to take into account the coherence of the square sequence, and this is reflected in the complexity of the coherence conditions between the models which form part of the forcing conditions. An advantage of this type of approach over the morass-based approach is that it requires less from the ground model—for example Friedman's forcing only needs a weakening of CH in the ground model. We use the full CH together with $2^{\omega_1} = \omega_2$. The main difficulties of both approaches of course are the same, and they stem from the fact that combinatorics at ω_2 is much less prone to independence than the combinatorics at ω_1, as exemplified by the above mentioned result of Shelah on $\diamondsuit$ [12]. It is both in developing combinatorics and fine forcing techniques that we can better understand the truth about ω_2. An interesting unified approach to adding objects to ω_2 is being developed by Neeman as well as Veličković and Venturi, in works in progress.

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2. Preliminaries

Most of the notation is standard. The relation $A \subset B$ means that A is either a proper subset of B or equal to B. $|X|$ is the cardinality of the set X. For a set of ordinals X, a limit point of X is an ordinal α such that $\alpha = \sup(Y)$ for some $Y \subset X$ or, equivalently, if $\alpha = \sup(X \cap \alpha)$. $\mathrm{Lim}(X)$ is the set of limit points of X. For a function f, $\mathcal{D}_f$ denotes the domain of f, and $f \restriction A$ denotes the restriction of f to the set $A \cap \mathcal{D}_f$. If α and β are ordinals then the interval (α, β) denotes the set $\{\mu \mid \mu \text{ is an ordinal}, \alpha < \mu < \beta\} = \beta \setminus (\alpha + 1)$. Closed and half open intervals are defined similarly. $[A]^\kappa$ is the set of all subsets of A of cardinality κ. The set $[A]^{\leqslant\kappa}$ is defined analogously.

For a regular cardinal θ, H_θ is the set of all sets x with hereditary cardinality less than θ (i.e. the transitive closure of x has cardinality less than θ). For $\theta > \omega_2$ we consider H_θ to be a model with the standard relation $\in$ and a fixed well-ordering $\leqslant^*$ and we write H_θ for the structure $(H_\theta, \in, \leqslant^*)$. We will primarily work with H_{ω_2} which we view as a model with $\in$ and $\leqslant^* \restriction H_{\omega_2}$. A cardinal θ is said to be *large enough* if every set in consideration is an element of H_θ.

Definition 2.1. Suppose κ is a regular cardinal. A set $C \subset \kappa$ is called a *closed unbounded* set or a *club* in κ if:

(1) for every $\lambda < \kappa$ and an increasing sequence $\langle \alpha_i \mid i < \lambda \rangle$ of elements from C, we have that $\bigcup_{i<\lambda} \alpha_i \in C$ (*closed*);
(2) for every $\alpha < \kappa$ there exists some $\beta \in C$ such that $\beta > \alpha$ (*unbounded*).

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