Annals of Pure and Applied Logic 165 (2014) 26–48

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

What is relevance logic?

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ARTICLE INFO

Article history:
Available online 20 August 2013

MSC:
03B22
03B47
03B53
03F52

Keywords:
Relevance logics
Entailment relations
Connectives
Linear logic

ABSTRACT

We suggest two precise abstract definitions of the notion of 'relevance logic' which are both independent of any proof system or semantics. We show that according to the simpler one, $\mathbf{R}_{\overrightarrow{\neg}}$ (the intensional fragment of $\mathbf{R}$) is the minimal relevance logic, but $\mathbf{R}$ itself is not. In contrast, $\mathbf{R}$ and many other logics are relevance logics according to the second (more complicated) definition, while all fragments of linear logic are not.

1. Introduction

Relevance logic is an extensive area of research, to which a huge number of books and articles have been devoted. The number of logics which have been investigated as "relevance logics" is very big too. Still, the literature on relevance logics seems to lack a precise definition or characterization of the notion of relevance logic.[1] Neither does it provide (as far as the author of this paper can tell) natural criteria according to which the systems which are classified as such in the various surveys of relevance logic (like [1,2,11,14]) deserve this name, while e.g. Girard's linear logic [12] does not.[2,3]

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[1] Usually surveys of this topic say at the beginning that a relevance logic is a logic in which a sentence ψ follows from a finite set of assumptions iff there is a proof of ψ from that set in which all the elements of this set are used. However, this cannot serve as it is as a precise criterion, because it depends on the concepts of 'proof' and 'use of an assumption in a proof', and neither of which is defined in the relevant literature in a coherent way. See [5] for further details and discussions.

[2] The best-known criterion used in the literature is the variable-sharing property (various variants of which indeed play a crucial role in this paper as well). However, linear logic satisfies this criterion too. Besides, so far the criterion has been formulated in the literature only in connection with some 'implication', and so accepting it makes sense only if the connective with which it is associated deserves being seen as an implication. Therefore in any case variable-sharing for itself is not sufficient. One should at least add some characterization of 'an implication connective'.

[3] Linear logic *is* sometimes officially included in the family of relevance logics. However, in practice researchers do not take it as such. Thus [11] devotes only half a page to it (out of 120), in which it is said that "linear logic is one which has a number of *formal similarities* to relevance logics" (the emphasis on 'formal similarities' is mine). Also the extensive two volumes [1,2] devote just half a page to linear logic, while [14] does not mention it at all.

0168-0072/\$ – see front matter
http://dx.doi.org/10.1016/j.apal.2013.07.004

In this paper we suggest two precise possible answers to the question what is propositional relevance logic. Both of our definitions are independent of any proof system or semantics. We believe that the first of them is very natural, and is perhaps what *should* be taken as the correct definition. Unfortunately, its scope is rather limited: it includes only systems that can be obtained from the logic $\mathbf{R}_{\overset{\neg}{\rightarrow}}$ by the addition of axiom schemas (but of course not all such systems). Thus even the system **R** (which is taken in [11] as the central relevance logic) is not a relevance logic according to that rather strict definition. This is why we suggest a second, more liberal, definition, under which **R** does fall, but not logics like classical logic, intuitionistic logic, the semi-relevant logic **RM**, or even linear logic and its fragments.

2. Logics and their properties

In the sequel $\mathcal{L}$ denotes a propositional language. The set of well-formed formulas of $\mathcal{L}$ is denoted by $\mathcal{W}(\mathcal{L})$, and φ, ψ, θ vary over its elements. $\mathcal{T}, \mathcal{S}$ vary over theories of $\mathcal{L}$ (where by a 'theory' we simply mean here a subset of $\mathcal{W}(\mathcal{L})$), and Γ, Δ vary over *finite* sets (and sometimes multisets) of formulas.[4] We denote by $\mathsf{Atoms}(\varphi)$ ($\mathsf{Atoms}(\mathcal{T})$) the set of atomic formulas that appear in φ (in the formulas of $\mathcal{T}$).

In order to characterize relevance logics, we first have to clarify what we mean here by a (propositional) *logic*. The definition we adopt is based on the notion of consequence relation, which was introduced by Tarski in [17]:

Definition 2.1. A (Tarskian) *consequence relation* (tcr) for a language $\mathcal{L}$ is a binary relation $\vdash$ between theories in $\mathcal{W}(\mathcal{L})$ and formulas in $\mathcal{W}(\mathcal{L})$, satisfying the following three conditions:

[R]	*Reflexivity*:	$\psi \vdash \psi$ (i.e. $\{\psi\} \vdash \psi$).
[M]	*Monotonicity*:	if $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}' \vdash \psi$.
[C]	*Cut* (*Transitivity*):	if $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \varphi$ then $\mathcal{T} \cup \mathcal{T}' \vdash \varphi$.

Definition 2.2. Let $\vdash$ be a Tarskian consequence relation for $\mathcal{L}$.

- $\vdash$ is *structural*, if for every $\mathcal{L}$-substitution θ and every $\mathcal{T}$ and ψ, if $\mathcal{T} \vdash \psi$ then $\theta(\mathcal{T}) \vdash \theta(\psi)$.
- $\vdash$ is *consistent* (or *non-trivial*) if $p \nvdash q$ for distinct atoms $p, q \in \mathsf{Atoms}(\mathcal{L})$.
- $\vdash$ is *finitary*, if for every theory $\mathcal{T}$ and every formula ψ such that $\mathcal{T} \vdash \psi$ there is a *finite* theory $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \vdash \psi$.

Tarski's notion of a consequence relation was generalized by Scott [16] to the multiple-conclusioned case:

Definition 2.3. A *consequence relation* in the sense of Scott (an scr) for a language $\mathcal{L}$ is a binary relation $\vdash$ between theories in $\mathcal{W}(\mathcal{L})$, satisfying the following three conditions:

[R]	*Reflexivity*:	$\psi \vdash \psi$ (i.e. $\{\psi\} \vdash \{\psi\}$).
[M]	*Monotonicity*:	if $\mathcal{T} \vdash \mathcal{S}$ and $\mathcal{T} \subseteq \mathcal{T}'$, $\mathcal{S} \subseteq \mathcal{S}'$, then $\mathcal{T}' \vdash \mathcal{S}'$.
[C]	*Cut* (*Transitivity*):	if $\mathcal{T} \vdash \psi, \mathcal{S}$ and $\mathcal{T}', \psi \vdash \mathcal{S}'$ then $\mathcal{T} \cup \mathcal{T}' \vdash \mathcal{S} \cup \mathcal{S}'$.

Structural, non-trivial, and finitary Scott consequence relations are defined by the obvious generalizations of the definitions in the single-conclusion case.

Now we define the notion of propositional logic which is used in this paper.

[4] Wherever it matters, it will be clear from the context whether we talk about finite sets or finite multisets.