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Possible world semantics for first-order logic of proofs

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ABSTRACT

In the tech report Artemov and Yavorskaya (Sidon) (2011) [4] an elegant formulation of the first-order logic of proofs was given, FOLP. This logic plays a fundamental role in providing an arithmetic semantics for first-order intuitionistic logic, as was shown. In particular, the tech report proved an arithmetic completeness theorem, and a realization theorem for FOLP. In this paper we provide a possible-world semantics for FOLP, based on the propositional semantics of Fitting (2005) [5]. We also give an Mkrtychev semantics. Motivation and intuition for FOLP can be found in Artemov and Yavorskaya (Sidon) (2011) [4], and are not fully discussed here.

This paper is dedicated to Sergei Artemov, an honored colleague and friend, who has made wonderful things for the rest of us to play with.

1. Introduction

Propositional Justification Logics are modal-like logics in which the usual necessity operator is split into a family of more complex terms called *justifications*. Instead of $\Box A$ one finds $t{:}A$, which can be read "t is a justification for A." The structure of t embodies, in a straightforward way, how we come to know A or verify A. Many standard propositional modal logics have justification logic counterparts, where the notion of *counterpart* has a precise definition via what are called *Realization Theorems*. One can think of justification logics as *explicit* versions of modal logics, with conventional modal operators embodying justifications in an *implicit* way, but we do not go further into this point here. The first propositional justification logic was LP, the Logic of Proofs, an explicit version of propositional S4. It was introduced by Artemov as part of a project to provide an arithmetic semantics for propositional intuitionistic logic, [1,2]. Briefly, propositional intuitionistic logic embeds into propositional S4 via the well-known Gödel translation. Propositional S4 in turn embeds into LP via a Realization Theorem. Propositional LP embeds into arithmetic, Artemov's *Arithmetical Completeness Theorem*, [1].

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Since this initial work there has been much study of propositional justification logics, thinking of them as explicit logics of knowledge or belief. A general survey of the subject can be found in [3]. But to reiterate, all this was at the propositional level. Recently Artemov and Yavorskaya [4] defined a first-order extension of the logic of proofs, FOLP. The original results on an arithmetic semantics for propositional intuitionistic logic were shown to extend to the first-order case as well. This completed the arithmetic semantics project for intuitionistic logic, but it also introduced a new family of interesting explicit logics to study.

In [5] a possible world semantics was introduced for LP, and for a few other propositional justification logics. (This has since been extended to a broad range of justification logics.) On the one hand this semantics elaborates the familiar Kripke semantics for modal logics by adding machinery to model the behavior of explicit reasons, and on the other hand it extends, in a direct way, an earlier LP semantics of Mkrtychev, [7]. The purpose of the present paper is to extend this propositional work to a first-order setting. The resulting possible world semantics obeys a *monotonicity* condition, familiar from propositional modal logics. This is natural because of the intended application to intuitionistic logic. We postpone to future work the study of constant domain versions. The work here is specifically for the first-order version of LP. Simple modifications adapt the results to several other logics, and we will discuss this briefly at the end of the paper.

The contents of this paper first appeared, in a somewhat different form, in [6].

2. FOLP, the language and the axioms

Let us think proof-theoretically for a bit. In a first-order proof free variables play two different but easily confused roles. One role is simply that of a formal symbol. The Universal Generalization rule allows us to claim a proof of $(\forall x)A(x)$ given a proof of $A(x)$. In this x is a syntactic object, with no inherent meaning. The other role of variables is that of a place-holder that can be substituted for. Suppose we have a proof of $A(x)$, and say 3 is a constant symbol of our language. We can turn the proof of $A(x)$ into a proof of $A(3)$ by going through it and replacing all free occurrences of x with occurrences of 3 (assuming universal generalization on x was never used). Similarly for $A(4)$ and so on. We can think of the proof as more like a proof template from which we can stamp out many concrete proofs. Note that we had to put in a *caveat* about non-use of universal generalization—the two roles of variables are not compatible.

In propositional LP, if t is a proof term (justification term) and A is a formula then $t{:}A$ is a formula, and can be thought of as asserting that A is so, with t as its proof (justification). Indeed, under an arithmetic interpretation, proof terms are interpreted as Gödel numbers for proofs. In the first-order language of [4] this is modified into $t{:}_X A$, where X is a finite set of variables. This can be thought of as asserting that t represents a proof of A in which the variables in X are the ones that can be substituted for, and hence are not allowed in applications of universal generalization. Our possible world semantics directly incorporates the idea of two roles for variables, as will be seen later on, and the axioms in this section should be read with the double role in mind.

This following language definition is taken from the technical report of Artemov and Yavorskaya, [4].

The language of FOLP has a countable set of predicate symbols of any arity, but no function symbols or equality. There are also countably many individual variables, and an atomic formula is $Q(x_1, x_2, \ldots, x_n)$, where Q is a predicate symbol of arity n and each x_i is an individual variable. (Typically we write $x, y, \ldots,$ with or without subscripts, for individual variables.) Formulas are built up as usual using Boolean connectives and quantifiers over individual variables, and one additional construction described below.

The language of FOLP also has a family of *proof terms*, more generally called *justification terms* when logics not directly connected with intuitionistic logic are considered. These are built up from a countable family of proof variables (typically p or p_i) and a countable family of proof constants (typically c or c_i). More complex proof terms are built up using special function and operation symbols, as follows. If t and s are proof terms, then $t \cdot s$, $!t$, and $t + s$ are proof terms. This much is inherited from the propositional logic LP, and we do not discuss their intended meanings here. [3] is a good reference. In addition, if x is an