Annals of Pure and Applied Logic 165 (2014) 316–339

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Relativized ordinal analysis: The case of Power Kripke–Platek set theory

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ARTICLE INFO

Article history:
Available online 19 August 2013

MSC:
primary 03F15, 03F05, 03F35
secondary 03F03

Keywords:
Power Kripke–Platek set theory
Ordinal analysis
Ordinal representation systems
Proof-theoretic strength
Power-admissible set

ABSTRACT

The paper relativizes the method of ordinal analysis developed for Kripke–Platek set theory to theories which have the power set axiom. We show that it is possible to use this technique to extract information about Power Kripke–Platek set theory, $\mathbf{KP}(\mathcal{P})$.

As an application it is shown that whenever $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$ proves a $\Pi_2^{\mathcal{P}}$ statement then it holds true in the segment V_τ of the von Neumann hierarchy, where τ stands for the Bachmann–Howard ordinal.

1. Introduction

Ordinal analyses of ever stronger theories have been obtained over the last 20 years (cf. [1–3,16,17,20–22,24,25]). The strongest theories for which proof-theoretic ordinals have been determined are subsystems of second-order arithmetic with comprehension restricted to Π_2^1-comprehension (or even Δ_3^1-comprehension). Thus it appears that it is currently impossible to furnish an ordinal analysis of any set theory which has the power set axiom among its axioms as such a theory would dwarf the strength of second-order arithmetic. Notwithstanding the foregoing, the current paper relativizes the techniques of ordinal analysis developed for Kripke–Platek set theory, $\mathbf{KP}$, to obtain useful information about Power Kripke–Platek set theory, $\mathbf{KP}(\mathcal{P})$, culminating in a bound for the transfinite iterations of the power set operation that are provable in the latter theory. It is perhaps worthwhile comparing the results in this paper with other approaches to relativizing the ordinal analysis of $\mathbf{KP}$. T. Arai [4] has used an ordinal representation system of Bachmann–Howard type enriched by Skolem functions to provide an analysis of Zermelo–Fraenkel set theory. In the approach of the present paper the ordinal representation is not changed at all. Rather than obtaining a characterization of the proof-theoretic ordinal of $\mathbf{KP}(\mathcal{P})$, we characterize the smallest segment of the von Neumann hierarchy which is closed under the provable power-recursive functions of $\mathbf{KP}(\mathcal{P})$ whereby one also obtains a proof-theoretic

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0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2013.07.016

reduction of $\mathbf{KP}(\mathcal{P})$ to Zermelo set theory plus iterations of the powerset operation to any ordinal below the Bachmann–Howard ordinal.[1] The same bound also holds for the theory $\mathbf{KP}(\mathcal{P})+\mathbf{AC}$, where **AC** stands for the *axiom of choice*. These theorems considerably sharpen results of H. Friedman to the extent that $\mathbf{KP}(\mathcal{P})+\mathbf{AC}$ does not prove the existence of the first non-recursive ordinal ω_1^{CK} (cf. [8, Theorem 2.5] and [13, Theorem 10]).

Technically we draw on tools that have been developed more than 30 years ago. With the pioneering work of Jäger [10] on Kripke–Platek set theory and its extensions to stronger theories by Jäger and Pohlers [11] the forum of ordinal analysis switched from subsystems of second-order arithmetic to set theory, shaping what is called *admissible proof theory*, after the standard models of **KP**. We also draw on the framework of operator controlled derivations developed by Buchholz [19] that allows one to express the uniformity of infinite derivations and to carry out their bookkeeping in an elegant way.

The results and techniques of this paper have important applications. The characterization of the strength of $\mathbf{KP}(\mathcal{P})$ in terms of the von Neumann hierarchy is used in [28, Theorem 1.1] to calibrate the strength of the calculus of construction with one type universe (which is an intuitionistic type theory). Another application is made in connection with the so-called *existence property*, **EP**, that intuitionistic set theories may or may not have. Full intuitionistic Zermelo–Fraenkel set theory, **IZF**, does not have the existence property, where **IZF** is formulated with Collection (cf. [9]). By contrast, an ordinal analysis of intuitionistic $\mathbf{KP}(\mathcal{P})$ similar to the one given in this paper together with results from [27] can be utilized to show that **IZF** with only bounded separation has the **EP**.

2. Power Kripke–Platek set theory

A particularly interesting (classical) subtheory of **ZF** is Kripke–Platek set theory, **KP**. Its standard models are called *admissible sets*. One of the reasons that this is an important theory is that a great deal of set theory requires only the axioms of **KP**. An even more important reason is that admissible sets have been a major source of interaction between model theory, recursion theory and set theory (cf. [5]). Roughly **KP** arises from **ZF** by completely omitting the power set axiom and restricting separation and collection to set bounded formulae but adding set induction (or class foundation). These alterations are suggested by the informal notion of 'predicative'.

To be more precise, quantifiers of the forms $\forall x \in a$, $\exists x \in a$ are called *set bounded*. *Set bounded* or Δ_0*-formulae* are formulae wherein all quantifiers are set bounded. The axioms of **KP** consist of *Extensionality, Pair, Union, Infinity,* Δ_0*-Separation*

$$\exists x\,\forall u\big[u \in x \leftrightarrow \big(u \in a \wedge A(u)\big)\big]$$

for all Δ_0-formulae $A(u)$, Δ_0-*Collection*

$$\forall x \in a\,\exists y G(x,y) \rightarrow \exists z\,\forall x \in a\,\exists y \in z G(x,y)$$

for all Δ_0-formulae $G(x,y)$, and *Set Induction*

$$\forall x\,\big[\big(\forall y \in x C(y)\big) \rightarrow C(x)\big] \rightarrow \forall x C(x)$$

for all formulae $C(x)$.

[1] The theories share the same $\Sigma_1^{\mathcal{P}}$ theorems, but are still distinct since Zermelo set theory does not prove $\Delta_0^{\mathcal{P}}$-Collection whereas $\mathbf{KP}(\mathcal{P})$ does not prove full Separation.