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# Modeling linear logic with implicit functions

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ABSTRACT

Just as intuitionistic proofs can be modeled by functions, linear logic proofs, being symmetric in the inputs and outputs, can be modeled by relations (for example, cliques in coherence spaces). However generic relations do not establish any functional dependence between the arguments, and therefore it is questionable whether they can be thought as reasonable generalizations of functions. On the other hand, in some situations (typically in differential calculus) one can speak in some precise sense about an implicit functional dependence defined by a relation. It turns out that it is possible to model linear logic with implicit functions rather than general relations, an adequate language for such a semantics being (elementary) differential calculus. This results in a non-degenerate model enjoying quite strong completeness properties.



## 1. Introduction

Linear logic (**LL**), introduced by J.-Y. Girard in the late eighties [10], has become an extremely popular subject. One of the attractive features of this system consists in combining its constructive nature (a possibility of functional interpretation of proofs), typical for intuitionistic logic, with the familiar symmetries of classical logic, such as the involutivity of negation and De Morgan dualities between connectives.

From the constructive point of view, a proof should be understood as a function, or, in more modern and general terms, a morphism, that can be composed with other proofs. Typically, proofs of the implications $A \to B$ and $B \to C$ can be composed to yield a proof of $A \to C$ (the rule of syllogism). Thus one can think of a category, whose objects are formulas, and whose morphisms are equivalence classes of proofs. In other words, one assumes existence of an equivalence relation on proofs that turns the set of proofs and formulas into a well-defined category. However such a functional interpretation is non-trivial only if there exist hom-sets with *more than one element*, in other words if there exist formulas with several non-equivalent proofs.

This is the case for intuitionistic logic, whose proofs can indeed be interpreted as functions (say, $\lambda$-terms). In this sense, intuitionistic logic is constructive, in fact a prototype of a constructive logic. Whereas classical logic admits only a degenerate categorical interpretation, defined by declaring all proofs of the same formula

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equivalent [14], pp. 67–116. (The corresponding category is the Boolean algebra of provably equivalent formulas.) On the other hand, classical logic enjoys a number of attractive symmetries, such as the duality between connectives, the involutive negation, the law of excluded middle — all of which are lost in the intuitionistic case.

Linear logic combines the constructive nature of intuitionistic logic and the symmetries of classical logic. This is achieved by taking control over unlimited use of hypotheses. In **LL** each hypothesis in the proof should be used only once and exactly once. (In this paper we discuss only the so-called *multiplicative* fragment of linear logic (**MLL**). The interested reader can find an introduction to the full linear logic for example in [12].)

### *1.1. Denotational semantics*

A *denotational model* of a constructive logic is a category where one can interpret formulas as objects and proofs as morphisms, preserving the internal categorical structure of the logic (connectives, rules, axioms, etc.). Finding denotational models is the problem of *denotational semantics.*

Intuitionistic logic, for example, can be interpreted simply in the category of sets and functions (although this is not a best model). Thus intuitionistic proofs, seen as $\lambda$-terms, represent "general" functions in quite a literal sense. On the other hand a functional explanation of linear logic is not completely obvious. Linear logic proofs are symmetric in the input and the output, and general functions are not. Thus linear logic proofs may correspond only to very special functions (such as linear operators) or to something more general than "general" functions. Thinking of relations as natural generalizations of functions, one often interprets linear logic proofs as relations. (This tradition goes back to Girard's work on quantitative semantics [11]. In such a semantics sets play the role of bases of vector spaces, and relations are analogous to matrices. In this paper we take somewhat more primitive view of relations, not anticipating any analogies with linear algebra.)

We note that **LL** cannot be characterized as the "general" logic of relations, the relational interpretation being very degenerate. Such an interpretation fails to capture much of the structure of **LL**, and perhaps this can be explained as follows. Linear logic proofs mix inputs and outputs indefinitely, and, thus, hide the correspondence between them. However such a correspondence is always present implicitly — for example in the form of identity links connecting dual literals in a proof-net. On the other hand a general relation does not imply any dependence between the arguments. A relation, coming from an actual **LL**-proof, always has the form of an *implicit function* — some of the arguments can be expressed as functions of the remaining ones.

This observation suggests the idea of modeling **LL** by means of implicit functions, typically in the setting of differential calculus, where a relevant theory is well developed. In differential geometry, relations, defining implicit functions, are supported at smooth submanifolds. Motivated by the above arguments, we develop a special relational interpretation of (multiplicative) linear logic, where proofs are modeled by *smooth* relations, i.e. by smooth submanifolds.

Such an interpretation however does not come for free from the usual relational semantics. One should specify the target category for the interpretation, and this is not completely trivial. An important phenomenon arising in the smooth setting is that smooth relations do not compose in general, i.e. the set-theoretic composition of smooth relations may fail to be smooth. In other words, smooth relations themselves do not form a category.

In order to get a well-defined denotational model, we interpret formulas as spaces (vector spaces or differentiable manifolds), equipped with a certain extra structure that we call the *smooth coherence space* structure, since, in some sense, it looks like a "smoothing" of the familiar coherence space structure of Girard. The extra structure (technically, two conic subsets of tangent/cotangent vectors) plays the role of