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Self-referentiality of Brouwer–Heyting–Kolmogorov semantics

Junhua Yu

Doctoral Program in Computer Science, The Graduate Center, City University of New York, 365 Fifth Avenue, New York, NY 10016-4309, USA

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The Gödel–Artemov framework offered a formalization of the Brouwer–Heyting–Kolmogorov (BHK) semantics of intuitionistic logic via classical proofs. In this framework, the intuitionistic propositional logic IPC is embedded in the modal logic S4, S4 is realized in the Logic of Proofs LP, and LP has a provability interpretation in Peano Arithmetic. Self-referential LP-formulas of the type 't is a proof of a formula ϕ containing t itself' are permitted in the realization of S4 in LP, and if such formulas are indeed involved, it is then necessary to use fixed-point arithmetical methods to explain intuitionistic logic via provability. The natural question of whether self-referentiality can be avoided in realization of S4 was answered negatively by Kuznets who provided an S4-theorem that cannot be realized without using directly self-referential LP-formulas. This paper studies the question of whether IPC can be embedded in S4 and then realized in LP without using self-referential formulas. We consider a general class of Gödel-style modal embeddings of IPC in S4 and by extending Kuznets' method, show that there are IPC-theorems such that, under each such embedding, are mapped to S4-theorems that cannot be realized in LP without using directly self-referential formulas. Interestingly, all double-negations of tautologies that are not IPC-theorems, like $\neg\neg(\neg\neg p \to p)$, are shown to require direct self-referentiality. Another example is found in $\mathsf{IPC}_{\to}$, the purely implicational fragment of IPC. This suggests that the BHK semantics of intuitionistic logic (even of intuitionistic implication) is intrinsically self-referential.
This paper is an extended version of [26].

1. Introduction

The Brouwer–Heyting–Kolmogorov semantics (BHK semantics) of Intuitionistic Propositional Logic IPC follows the reading of intuitionistic truth as provability. This was initially suggested by Brouwer and then stipulated informally by Heyting and Kolmogorov. Typically,

- $\perp$ has no proof, and
- a proof of $\phi \to \psi$ is a construction that returns a proof of ψ whenever a proof of ϕ is given.

E-mail address: jyu1@gc.cuny.edu.

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Gödel [13] contributed to this by introducing a modal calculus for provability that is essentially S4, and suggesting a possible embedding of IPC in his calculus. The suggested embedding, prefixing each subformula with the provability modality $\Box$, reflects the intuitionistic view of logical truth as provability. (To be exact, in [13], Gödel suggested the embedding "prefixing each subformula with a $\Box$" along with some of its S4-equivalent simplifications. We will consider Gödel-style modal embeddings in a general setting and get results that are not sensitive to the choice of embedding.) Gödel's embedding of IPC in S4 was shown to be faithful by McKinsey and Tarski [19] and hence transformed the problem of finding a provability semantics for IPC to finding a provability semantics for S4. Artemov [1,2] filled the gap by introducing the Logic of Proofs LP with completeness w.r.t. formal arithmetical provability and establishing an embedding, called *realization*, of S4 in LP. A detailed description of the approach of formalizing BHK semantics of IPC can be found in [2].

In this work, we take $\bot$, $\to$ as primitive connectives of classical propositional language, and $\bot$, $\wedge$, $\vee$, $\to$ as that of intuitionistic propositional language. Connectives like $\neg$, $\wedge$, $\vee$ are defined in the standard way where they are not primitive. In (classical) modal language, we take $\Box$ as the only primitive modality, and treat $\Diamond$ as an abbreviation of $\neg\Box\neg$. By a *prime* formula, we mean a propositional atom, or a $\bot$. Formulas are denoted by Greek letters. For *polarities of subformulas*: each formula is a positive subformula of itself; all negative (positive) subformulas of α and all positive (negative) subformulas of β are positive (negative) subformulas of $\alpha \to \beta$; the polarity of a subformula of α in $\Box\alpha$ is the same as that in α. If ψ is a negative (positive) subformula of ϕ, then we say that ψ *occurs negatively (positively)* in ϕ, and also, that the main connective of ψ (if any) is a *negative (positive) connective* in ϕ. For binary relations R and R' (defined as sets), by $R(x,y)$ we mean the pair $\langle x,y\rangle$ is in R, by $\overline{R}(x,y)$ we mean the pair $\langle x,y\rangle$ is not in R, and by $R \setminus R'(x,y)$ we mean $R(x,y)$ and $\overline{R'}(x,y)$.

We start with the definition of LP.

Definition 1 *(The Logic of Proofs* LP*).* (See [2].) In the language of LP: *formulas* are defined by $\phi ::= \bot \,|\, p \,|\, \phi \to \phi \,|\, t : \phi$, where $t ::= c \,|\, x \,|\, t \cdot t \,|\, t + t \,|\, !t$ is called a *term*, c is a *constant*, x is a (proof) *variable*, and p is an (propositional) atom.

LP has the following axiom schemes

(A0) A finite complete set of classical propositional axiom schemes,
(A1) $t{:}\phi \to \phi$,
(A2) $t_1{:}(\phi \to \psi) \to (t_2{:}\phi \to t_1{\cdot}t_2{:}\psi)$,
(A3) $t{:}\phi \to !t{:}t{:}\phi$,
(A4) $t_1{:}\phi \to t_1 + t_2 : \phi$ and $t_2{:}\phi \to t_1 + t_2{:}\phi$,

and rules

(MP) Modus Ponens,
(AN) $\dfrac{}{c : A}$ where c is a constant and A is an axiom.

A *constant specification*, denoted by $\mathcal{CS}$, is a set of formulas of the form $c{:}A$ where c is a constant and A is an axiom. By LP($\mathcal{CS}$), we mean the fragment of LP with only formulas from $\mathcal{CS}$ being allowed by rule (AN).

Each LP-proof (derivation) *calls for* a constant specification, namely the set of formulas introduced by rule (AN) in this proof (derivation). Clearly, an LP-proof (derivation) that calls for $\mathcal{CS}$ is an LP($\mathcal{CS}$)-proof (derivation).