Annals of Pure and Applied Logic 164 (2013) 1101–1117

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Axiomatizing first-order consequences in dependence logic

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ARTICLE INFO

Article history:
Received 29 October 2011
Accepted 13 March 2013
Available online 6 June 2013

MSC:
03C80
03C85
03B15
68P15

Keywords:
Dependence logic
Game logic
Independence logic
Independence friendly logic
Natural deduction

ABSTRACT

Dependence logic, introduced in Väänänen (2007) [11], cannot be axiomatized. However, first-order consequences of dependence logic sentences can be axiomatized, and this is what we shall do in this paper. We give an explicit axiomatization and prove the respective Completeness Theorem.

1. Introduction

Dependence logic was introduced in [11]. It extends ordinary first-order logic by new atomic formulas $=(x_1, \ldots, x_n, y)$ with the intuitive meaning that the values of the variables $x_1, \ldots, x_n$ completely determine the value of y is. This means that the relevant semantic game is a game of imperfect information. A player who picks y and claims that her strategy is a winning strategy should make the choice so that if the strategy is played twice, with the same values for $x_1, \ldots, x_n$, then the value of y is the same as well. Dependence logic cannot be axiomatized, for the set of its valid formulas is of the same complexity as that of full second-order logic. However, the first-order consequences of dependence logic sentences can be axiomatized. In this paper we give such an axiomatization.

Let us quickly review the reason why dependence logic cannot be effectively axiomatized. Consider the sentence

$$\theta_1\colon\ \exists z \forall x \exists y \big(=(y, x) \wedge \neg y = z\big).$$

We give the necessary preliminaries about dependence logic in the next section, but let us for now accept that θ_1 is true in a model if and only if the domain of the model is infinite. The player who picks y has to pick a different y for different x. Although dependence logic does not have a negation in the sense of classical logic, the mere existence of θ_1 in dependence logic should give a hint that axiomatization is going to be a problem. Elaborating but a little, θ_1 can be turned into a

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1 Supported by grant 264917 of the Academy of Finland.
2 Research partially supported by grant 251557 of the Academy of Finland and by the EUROCORES LogICCC LINT programme.

0168-0072/$ – see front matter
http://dx.doi.org/10.1016/j.apal.2013.05.006

sentence θ_2 in the language of arithmetic which says that some elementary axioms of number theory fail or else some number has infinitely many predecessors. We can now prove that a first-order sentence ϕ of the language of arithmetic is true in $(\mathbb{N}, +, \times, <)$ if and only if $\theta_2 \vee \phi$ is logically valid (true in every model) in dependence logic. This can be seen as follows: Suppose first ϕ is true in $(\mathbb{N}, +, \times, <)$. Let us take an arbitrary model M of the language of arithmetic. If $M \models \theta_2$, we may conclude $M \models \theta_2 \vee \phi$. So let us assume $M \not\models \theta_2$. Thus M satisfies the chosen elementary axioms of number theory and every element has only finitely many predecessors. As a consequence, $M \cong (\mathbb{N}, +, \times, <)$, so $M \models \phi$, and again $M \models \theta_2 \vee \phi$. For the converse, suppose $\theta_2 \vee \phi$ is logically valid. Since $(\mathbb{N}, +, \times, <)$ fails to satisfy θ_2, we must conclude that ϕ is true in $(\mathbb{N}, +, \times, <)$.

The above inference demonstrates that truth in $(\mathbb{N}, +, \times, <)$ can be reduced to logical validity in dependence logic. Thus, by Tarski's Undefinability of Truth argument, logical validity in dependence logic is non-arithmetical, and there cannot be any (effective) complete axiomatization of dependence logic.

The negative result just discussed would seem to frustrate any attempt to axiomatize dependence logic. However, there are at least two possible remedies. The first is to modify the semantics—this in the line adopted in Henkin's Completeness Theorem for second-order logic. For dependence logic this direction is taken in Galliani [5]. The other remedy is to restrict to a fragment of dependence logic. This is the line of attack of this paper. We restrict to logical consequences $T \models \phi$, in which T is in dependence logic but ϕ is in first-order logic.

The advantage of restricting to $T \models \phi$, with first-order ϕ, is that we can reduce the Completeness Theorem, assuming that $T \cup \{\neg\phi\}$ is deductively consistent, to the problem of constructing a model for $T \cup \{\neg\phi\}$. Since dependence logic can be translated to existential second-order logic, the construction of a model for $T \cup \{\neg\phi\}$ can in principle be done in first-order logic, by translating T to first-order by using new predicate symbols. This observation already shows that $T \models \phi$, for first-order ϕ, can *in principle* be axiomatized. Our goal in this paper is to give an explicit axiomatization.

The importance of an *explicit* axiomatization over and above the mere knowledge that an axiomatization exists, is paramount. The axioms and rules that we introduce throw light in a concrete way on logically sound inferences concerning dependence concepts. It turns out, perhaps unexpectedly, that fairly simple albeit non-trivial axioms and rules suffice.

Our axioms and rules are based on Barwise [2], where approximations of Henkin sentences, sentences which start with a partially ordered quantifier, are introduced. The useful method introduced by Barwise builds on earlier work on game formulas by Svenonius [10] and Vaught [13].

By axiomatizing first-order consequences we get an axiomatization of inconsistent dependence logic theories as a bonus, contradiction being itself expressible in first-order logic. The possibility of axiomatizing inconsistency in IF logic—a relative of dependence logic—has been emphasized by Hintikka [7].

The structure of the paper is the following. After the preliminaries we present our system of natural deduction in Section 3. In Section 4 we give a rather detailed proof of the Soundness of our system, which is not a priori obvious. Section 5 is devoted to the proof, using game formulas and their approximations, of the Completeness Theorem. The final section gives examples and open problems.

The second author is indebted to John Burgess for suggesting the possible relevance for dependence logic of the work of Barwise on approximations of Henkin formulas.

2. Preliminaries

In this section we define Dependence Logic ($\mathcal{D}$) and recall some basic results about it.

Definition 1. (See [11].) The syntax of $\mathcal{D}$ extends the syntax of FO, defined in terms of $\vee$, $\wedge$, $\neg$, $\exists$ and $\forall$, by new atomic formulas (dependence atoms) of the form

$$=(t_1, \ldots, t_n), \tag{1}$$

where $t_1, \ldots, t_n$ are terms. For a vocabulary τ, $\mathcal{D}[\tau]$ denotes the set of τ-formulas of $\mathcal{D}$.

The intuitive meaning of the dependence atom (1) is that the value of the term t_n is functionally determined by the values of the terms $t_1, \ldots, t_{n-1}$. As singular cases we have $=()$ which we take to be universally true, and $=(t)$ meaning that the value of t is constant.

The set $\mathrm{Fr}(\phi)$ of free variables of a formula $\phi \in \mathcal{D}$ is defined as for first-order logic, except that we have the new case

$$\mathrm{Fr}\big(=(t_1, \ldots, t_n)\big) = \mathrm{Var}(t_1) \cup \cdots \cup \mathrm{Var}(t_n),$$

where $\mathrm{Var}(t_i)$ is the set of variables occurring in the term t_i. If $\mathrm{Fr}(\phi) = \emptyset$, we call ϕ a sentence.

In order to define the semantics of $\mathcal{D}$, we first need to define the concept of a *team*. Let $\mathfrak{A}$ be a model with domain A. *Assignments* of $\mathfrak{A}$ are finite mappings from variables into A. The value of a term t in an assignment s is denoted by $t^{\mathfrak{A}}\langle s\rangle$. If s is an assignment, x a variable, and $a \in A$, then $s(a/x)$ denotes the assignment (with domain $\mathrm{Dom}(s) \cup \{x\}$) which agrees with s everywhere except that it maps x to a.

Let A be a set and $\{x_1, \ldots, x_k\}$ a finite (possibly empty) set of variables. A *team* X of A with domain $\mathrm{Dom}(X) = \{x_1, \ldots, x_k\}$ is any set of assignments from the variables $\{x_1, \ldots, x_k\}$ into the set A. We denote by $\mathrm{rel}(X)$ the k-ary relation of A corresponding to X