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A proof-theoretic account of classical principles of truth

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ABSTRACT

This paper explores the interface between principles of self-applicable truth and classical logic. To this end, the proof-theoretic strength of a number of axiomatic theories of truth over intuitionistic logic is determined. The theories considered correspond to the maximal consistent collections of fifteen truth-theoretic principles as isolated in Leigh and Rathjen (2012).

1. Introduction

In the analysis of the truth-theoretic paradoxes it is often the underlying logic that receives the blame. It is well known that in a purely classical environment, compositionality and self-reference do not mix well, so weaker logics have been suggested for the truth predicate. These are typically either partial or paraconsistent logics (for example [8,11]). Compared with classical logic however, such logics are not well understood and, as Feferman famously remarks, "nothing like sustained ordinary reasoning can be carried on in [them]" [7, p. 95]. This paper explores the interface between principles of self-applicable truth and classical logic more deeply, in particular the rôle classical principles play in restricting the free use of the truth predicate. To this end, we analyse various axiomatisations of truth over intuitionistic logic.

Intuitionistic logic has not received much attention to date from the truth theory community, but it does possess a number of virtues. As a logic upon which to study the effect of classical reasoning, it is better suited than other weakenings of classical logic because of its mature model theory and proof theory. This means that consistency and conservativeness results for truth over intuitionistic logic can be easily interpreted outside the field of theories of truth (for example in set theory or second-order analysis) as they can with classical logic. It also provides a firm base over which to study constructive interpretations of truth either in their own right or as approximations to classical truth.

We consider theories of truth that arise from expanding Heyting arithmetic by a fresh predicate symbol T and axioms regarding T taken from a collection of fifteen principles (these are listed in Table 1 below). The principles under consideration extend the twelve "Optional Axioms" isolated by Friedman and Sheard [9] and include various weakenings of the

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faulty truth bi-conditional $A \leftrightarrow \mathrm{T}\ulcorner A\urcorner$, axioms describing the compositional and self-applicable nature of truth and principles stating that the truth predicate has necessarily a complete or consistent interpretation. In [9], Friedman and Sheard showed that the twelve Optional Axioms give rise to nine maximal consistent theories of truth. Subsequent work by Cantini [6], Halbach [10] and Leigh and Rathjen [14] establish the proof-theoretic strength of each of the nine theories proving they range from conservative extensions of Peano arithmetic to the strength of one inductive definition, ID_1.

In [15] the same principles of truth are analysed over intuitionistic logic. It is observed that there are natural principles that, although equivalent over the classical base theory utilised in [9] and [14], can be separated over a purely intuitionistic base theory. As such, it is shown that some of the inconsistencies noted in [9] can be attributed to classical principles inherent in the base theory, the law of excluded middle or the statement that the logic under the truth predicate is classical, and that these sets of principles are consistent over the intuitionistic base theory. For instance, if the logic under the truth predicate is classical, the principles $\mathrm{T}\ulcorner A \vee B\urcorner \to \mathrm{T}\ulcorner A\urcorner \vee \mathrm{T}\ulcorner B\urcorner$ and $\mathrm{T}\ulcorner A\urcorner \vee \mathrm{T}\ulcorner \neg A\urcorner$ (stated for arbitrary sentences A, B) are equivalent. Nevertheless, there are models of the intuitionistic base theory that satisfy the former but refute the latter principle. In fact, the former axiom is consistent with all consistent sets of truth principles, while the latter is inconsistent with roughly half. The upshot is that while there are still exactly nine maximal consistent sets of principles over the new base theory, reverting to intuitionistic logic provides more freedom to express principles of truth while avoiding the pitfalls of inconsistency.

In this paper we analyse the nine maximal consistent theories of truth isolated in [15] and determine their proof-theoretic strength. The theories we analyse separate into two categories. Theories in the first group comprise all the compositional axioms for the logical symbols and so possess truth predicates that formalise satisfaction in certain classical or intuitionistic models. What distinguishes them from each other is the manner in which self-reference appears. This ranges from trivial instances of self-reference to finitely iterated truth akin to theories of ramified truth. The second group consists of those theories not containing the compositional axiom for implication. Instead there is a greater freedom to state self-referential axioms, in particular the axiom $\mathrm{T}\ulcorner \mathrm{T}\dot{x}\urcorner \to \mathrm{T}x$ and the rule of inference "from $\mathrm{T}\ulcorner A\urcorner$ infer A". As a result in these theories the truth predicate formalises a notion of idealised provability and intuitionistic truth.

1.1. Preliminaries

Let HA and PA be, respectively, the theories of Heyting and Peano arithmetic formulated in the language $\mathcal{L}$. We denote by $\mathcal{L}_{\mathrm{T}}$, the language $\mathcal{L}$ extended to include a unary predicate T. HA_{T} is HA formulated in $\mathcal{L}_{\mathrm{T}}$, that is with the schema of induction extended to formulæ of $\mathcal{L}_{\mathrm{T}}$, likewise PA_{T}. Formulæ of $\mathcal{L}$ are called arithmetical.

We fix a primitive recursive coding $\ulcorner . \urcorner$ of $\mathcal{L}_{\mathrm{T}}$ into the natural numbers along with suitable primitive recursive functions $\dot\wedge$, $\dot\vee$, and $\dot\to$ so that $\ulcorner A\urcorner \dot\circ \ulcorner B\urcorner = \ulcorner A \circ B\urcorner$ for all formulæ A and B with $\circ \in \{\wedge, \vee, \to\}$, and $m \dot\circ n = \ulcorner \bar{0} \neq \bar{0}\urcorner$ if either x or y is not the code of a formula. Also present is a function $sub(m,n)$ such that $sub(m,n) = \ulcorner A(\bar{n})\urcorner$ if $m = \ulcorner A(x)\urcorner$ where A has at most x free, and $sub(m,n) = \ulcorner \bar{0} = \bar{0}\urcorner$ otherwise. $\mathrm{Sent}_{\mathcal{L}_{\mathrm{T}}} x$ and $\mathrm{Prov}_{\mathsf{S}}\, x$ are formulæ expressing, respectively, that x is the code of a sentence in $\mathcal{L}_{\mathrm{T}}$, and x is the code of a formula provable in the theory S. We also make use of the usual abbreviations in this context including: $\forall\ulcorner A\urcorner B(\ulcorner A\urcorner)$ for $\forall z(\mathrm{Sent}_{\mathcal{L}_{\mathrm{T}}} z \to Bz)$ and $\forall\ulcorner A(x)\urcorner B(\ulcorner A\urcorner)$ for $\forall z(\mathrm{Sent}_{\mathcal{L}_{\mathrm{T}}}\, sub(z, \bar{0}) \to Bz)$ where z is some fresh variable symbol, and similarly for existential quantification; and $\ulcorner A(\dot{y})\urcorner$ to abbreviate the term $sub(\ulcorner A(x)\urcorner, y)$. SENT denotes the set of all codes of $\mathcal{L}_{\mathrm{T}}$-formulæ.

We fix a suitably large notation system for ordinals. We use Greek letters α, β etc. to range over ordinals in this system and, to reduce notation, the same symbols will also represent formal variables ranging over their arithmetical codes. Moreover, $<$ will be used for the induced less-than relation (assumed to be primitive recursive) on the notation system as well as the standard ordering on the natural numbers. Context should dictate which reading is appropriate.

For a formula $A(x)$, $\mathrm{TI}(A, x)$ denotes the axiom of *transfinite induction on A up to* (the ordinal encoded as) x, in symbols

$$\mathrm{Prog}\, A \to \forall \beta < x A(\beta),$$

where $\mathrm{Prog}\, A$ is the formula expressing that A is progressive in x,

$$\forall x\big((\forall \beta < x A(\beta)) \to A(x)\big).$$

For a language $\mathcal{L}_0$ and ordinal α, $\mathrm{TI}_{\mathcal{L}_0}(<\alpha)$ denotes the schema of transfinite induction on initial segments of α for $\mathcal{L}_0$, that is the collection of formulæ $\mathrm{TI}(A, \bar{m})$ where A is from $\mathcal{L}_0$ and m encodes an ordinal $\beta < \alpha$.

Let S and T be theories formulated in either $\mathcal{L}$ or $\mathcal{L}_{\mathrm{T}}$. Then S and T are said to be *proof-theoretically equivalent* if they prove the same theorems of $\mathcal{L}$ and, moreover, that this fact can be established within HA. If, moreover, S is formulated in $\mathcal{L}$ and T in $\mathcal{L}_{\mathrm{T}}$, we call T a *conservative extension* of S. The *proof-theoretic ordinal* of a theory is defined to be the smallest ordinal α (from our fixed notation system) such that the theory is proof-theoretically equivalent to (or a conservative extension of) either $\mathsf{HA} + \mathrm{TI}_{\mathcal{L}}(<\alpha)$ or $\mathsf{PA} + \mathrm{TI}_{\mathcal{L}}(<\alpha)$. As we will be only concerned with determining the proof-theoretic ordinal of (what turn out to be) consistent, predicatively reducible theories formulated in $\mathcal{L}_{\mathrm{T}}$, it suffices to pick a notation system for Γ_0.

We make extensive use of models of intuitionistic logic, in particular intuitionistic Kripke ω-structures for $\mathcal{L}_{\mathrm{T}}$, which are introduced below. $\mathbb{N}$ denotes the standard (classical) model of arithmetic.