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Towards a canonical classical natural deduction system

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A B S T R A C T

This paper studies a new classical natural deduction system, presented as a typed calculus named $\underline{\lambda}\mu$let. It is designed to be isomorphic to Curien and Herbelin's $\bar{\lambda}\mu\tilde{\mu}$-calculus, both at the level of proofs and reduction, and the isomorphism is based on the correct correspondence between cut (resp. left-introduction) in sequent calculus, and substitution (resp. elimination) in natural deduction. It is a combination of Parigot's $\lambda\mu$-calculus with the idea of "coercion calculus" due to Cervesato and Pfenning, accommodating let-expressions in a surprising way: they expand Parigot's syntactic class of named terms.

This calculus and the mentioned isomorphism Θ offer three missing components of the proof theory of classical logic: a canonical natural deduction system; a robust process of "read-back" of calculi in the sequent calculus format into natural deduction syntax; a formalization of the usual semantics of the $\bar{\lambda}\mu\tilde{\mu}$-calculus, that explains co-terms and cuts as, respectively, contexts and hole-filling instructions. $\underline{\lambda}\mu$let is not yet another classical calculus, but rather a canonical reflection in natural deduction of the impeccable treatment of classical logic by sequent calculus; and Θ provides the "read-back" map and the formalized semantics, based on the precise notions of context and "hole-expression" provided by $\underline{\lambda}\mu$let.

We use "read-back" to achieve a precise connection with Parigot's $\lambda\mu$, and to derive λ-calculi for call-by-value combining control and let-expressions in a logically founded way. Finally, the semantics Θ, when fully developed, can be inverted at each syntactic category. This development gives us license to see sequent calculus as the semantics of natural deduction; and uncovers a new syntactic concept in $\bar{\lambda}\mu\tilde{\mu}$ ("co-context"), with which one can give a new definition of η-reduction.

1. Introduction

In this paper we introduce a new natural deduction system $\underline{\lambda}\mu$let for classical logic, presented as an extension of Parigot's $\lambda\mu$-calculus [25], and equipped with let-expressions. The main design principle of $\underline{\lambda}\mu$let was to obtain a system *isomorphic* to the classical sequent calculus $\bar{\lambda}\mu\tilde{\mu}$ of Curien and Herbelin [7], in order to have, in the natural deduction side, a system as faithful to the classical dualities as $\bar{\lambda}\mu\tilde{\mu}$. For this reason, $\underline{\lambda}\mu$let is not yet another calculus for classical logic, but rather a canonical reflection in natural deduction of the impeccable treatment of classical logic by sequent calculus.

The design of $\underline{\lambda}\mu$let is not a mere formal achievement; we will try to prove that $\underline{\lambda}\mu$let is full of syntactic subtlety, semantic insight, and, mainly, that it is the appropriate tool to make progress simultaneously in three different but related areas: (i) semantics of $\bar{\lambda}\mu\tilde{\mu}$; (ii) CBV λ-calculus; (ii) natural deduction for classical logic. Before we explain $\underline{\lambda}\mu$let in more detail, we expand on the problems we will address in these three areas.

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Semantics of $\bar{\lambda}\mu\tilde{\mu}$. Gentzen [15] refined the de Morgan classical duality from the level of provability to the level of proofs, by defining the sequent calculus LK, a symmetric proof system for classical logic exhibiting a duality between hypothesis and conclusion. Recently Curien and Herbelin [7] introduced a variant of LK and the corresponding $\bar{\lambda}\mu\tilde{\mu}$-calculus, extending the Curry–Howard correspondence to classical sequent calculus, and showing that classical logic also contains a duality, at the level of cut elimination, between call-by-name (CBN) and call-by-value (CBV) computation.

Even if it is clear that $\bar{\lambda}\mu\tilde{\mu}$ is some sort of functional language with control facilities, its full understanding rests, so far, on intuitions that are vague and deserve to be formalized. We mean the explanation of co-terms as "contexts" (in particular, the $\tilde{\mu}$ operator is explained in terms of a let-expression with a hole); and the explanation of cuts as "hole filling" in those contexts [7,19]. These contexts are derived from some natural deduction syntax, some variant of $\lambda\mu$, extended with let-expressions. But which variant exactly? We prove the answer is $\underline{\lambda}\mu\mathsf{let}$. The contexts that interpret co-terms are a derived syntactic notion of $\underline{\lambda}\mu\mathsf{let}$; and filling the hole of such contexts results in expressions of a certain syntactic class of $\underline{\lambda}\mu\mathsf{let}$ named *statements*. This semantics is nothing less than the isomorphism $\Theta : \bar{\lambda}\mu\tilde{\mu} \to \underline{\lambda}\mu\mathsf{let}$.[1]

The CBV λ-calculus. Through $\bar{\lambda}\mu\tilde{\mu}$, Curien and Herbelin reduced the essence of the non-determinism in classical cut-elimination to a single critical pair, and recognized in this critical pair the choice between CBN and CBV computation. In particular, this opened the way to the definition of CBV fragments of $\bar{\lambda}\mu\tilde{\mu}$ and to a proof-theoretical answer to the question "what is CBV λ-calculus?", a question firstly posed in [26], and explicitly addressed in [7]. Such proof-theoretical approach contrasts with the developments in [22,32,33] that put forward Moggi's computational λ-calculus as the CBV λ-calculus.

For a variety of reasons, one would like to see the sequent calculus account of CBV translated to natural deduction. First, because that would provide a "read-back" [7,19] of $\bar{\lambda}\mu\tilde{\mu}$ proof expressions into a language where the familiar notation of functional application is available and, therefore, a language closer to actual programming languages. Second, because it is rather natural to ask whether the proof-theoretical understanding of CBV is an exclusive of sequent calculus, that is, whether natural deduction is, for some reason, doomed to account only for CBN computation. This read-back effort, already found in [7], continued through [19,30,20]. But the effort shows many difficulties. It is to a large extent informal, as the target system is not properly developed; it shows hesitations, as some attempts admittedly failed [19]; and it is even contradictory, as the definition of CBV $\lambda\mu$-calculus in [30] disagrees with that of [24].

We propose the isomorphism $\Theta : \bar{\lambda}\mu\tilde{\mu} \to \underline{\lambda}\mu\mathsf{let}$ as a systematic read-back process, with a fully formalized and developed natural deduction target. In particular, one obtains CBV λ-calculi in natural deduction format by restricting Θ to CBV fragments of $\bar{\lambda}\mu\tilde{\mu}$ and characterizing the range of such restrictions.

Natural deduction for classical logic. As we are seeing, the need to develop natural deduction for classical logic has many external motivations, but it also arises from the internal difficulties of the theory of natural deduction.

Both Gentzen [15] and Prawitz [28] defined natural deduction for classical logic as intuitionistic natural deduction supplemented with some classical inference principle. Prawitz admits that "this is perhaps not the most natural procedure from the classical logic point of view", as it does not reflect the de Morgan symmetry at the level of proofs ([28], p. 44); and Gentzen observed that there is no canonical choice as to what inference principle to add. Computationally, through the Curry–Howard correspondence, this means that the λ-calculus may be extended with a variety of control operators: for instance $\mathcal{C}$, Δ, or `call-cc`, corresponding to the principles double-negation elimination, *reductio ad absurdum*, and Peirce's law, respectively [14,17,29,1].

So, nothing like a canonical system is obtained through Gentzen–Prawitz approach to classical natural deduction. Moreover, the design difficulties are accompanied by technical problems. According to [35], Prawitz [28] proves only a "slightly weakened subformula property"; and restriction of *reductio ad absurdum* to atomic conclusions only works for some logical constants [28,34]. The latter problem is solved in [23] through the adoption of general elimination rules [36] and replacing *reductio ad absurdum* with another principle: elimination from excluded middle. A "structural" approach to overcome the mentioned difficulties is to move to systems whose deductions conclude not a single formula, but rather a list or set of formulas. This is already found in the system of Boričić [5], where an elegant subformula property holds. But the full computational power of the explicit manipulation of several conclusions is revealed by Parigot in his $\lambda\mu$-calculus [25].

Adopting the multiple conclusion framework allows us to depart from the intuitionistic system in a way that avoids having to chose among the variety of classical principles. But it still does not guarantee a natural deduction system faithful to the classical dualities. This is easy to verify: by Curien and Herbelin, the duality CBN/CBV is a duality of classical logic; but it is not clear even how to define a CBV variant of Parigot's $\lambda\mu$, as the distinct proposals [24,7,19,30,20] show, let alone give a natural deduction explanation of CBV based on $\lambda\mu$. One of the problems is that let-expressions are unavoidable in CBV λ-calculi, and therefore one has to offer some proof-theoretical understanding of them. In particular, natural deduction has to be extended. But how? This paper claims that let-expressions *cannot* be added to $\lambda\mu$ in a routine way, taking them as another form of proof-term; moreover, one has to understand the difference between the typing rule for let-expressions (which is a substitution inference rule) and the cut rule in sequent calculus. Cut and substitution are (perhaps in a subtle way) different, although linked by the isomorphism $\Theta : \bar{\lambda}\mu\tilde{\mu} \to \underline{\lambda}\mu\mathsf{let}$.

[1] When the reader sees $\underline{\lambda}\mu\mathsf{let}$ maybe (s)he will be surprised: how can such a system be the "explanation" of $\bar{\lambda}\mu\tilde{\mu}$? The measure of the reader's surprise is the measure of how vague and unclear were for her/him the explanations in terms of contexts and hole filling, and of how much the formalization of such explanations is needed.