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Continuation-passing style models complete for intuitionistic logic

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ABSTRACT

A class of models is presented, in the form of continuation monads polymorphic for first-order individuals, that is sound and complete for minimal intuitionistic predicate logic (including disjunction and the existential quantifier). The proofs of soundness and completeness are constructive and the computational content of their composition is, in particular, a β-normalisation-by-evaluation program for simply typed lambda calculus with sum types. Although the inspiration comes from Danvy's type-directed partial evaluator for the same lambda calculus, the use of delimited control operators (i.e. computational effects) is avoided. The role of polymorphism is crucial – dropping it allows one to obtain a notion of model complete for classical predicate logic.

1. Introduction

Although Kripke models are standard semantics for intuitionistic logic, there is as yet no (simple) constructive proof of their completeness when one considers all the logical connectives. While Kripke's original proof [25] was classical, Veldman gave an intuitionistic one [32] by using Brouwer's Fan Theorem to handle disjunction and the existential quantifier. To see what the computational content behind Veldman's proof is, one might consider a realisability interpretation of the Fan Theorem (for example [5]), but, all known realisers being defined by general recursion, due to the absence of an elementary proof of their termination, it is not clear whether one can think of the program using them as a constructive proof or not.

On the other hand, a connection between normalisation-by-evaluation (NBE) [6] for simply typed lambda calculus, $\lambda^{\to}$, and completeness of minimal intuitionistic logic for Kripke models for the fragment $\{\wedge, \Rightarrow, \forall\}$ has been made [8,19]. We review this connection in Section 2. There, we also revisit Danvy's extension [10] of NBE from $\lambda^{\to}$ to $\lambda^{\to\vee}$, simply typed lambda calculus with sum types. Even though Danvy's algorithm is simple and elegant, he uses the full power of delimited control operators which do not yet have a typing system that permits to understand that use logically. We deal with that problem in Section 3, by modifying the notion of Kripke model so that we can give a proof of completeness for full intuitionistic logic in continuation-passing style, that is, without relying on having delimited control operators in our meta-language. In Section 4, we extract the algorithm behind the given completeness proof, a β-NBE algorithm for $\lambda^{\to\vee}$. In Section 5, we stress the importance of our models being parametric, by comparing them to similar models that are complete for classical logic [23]. We conclude with Section 6, where we also mention related work.

The proofs of Section 3 have been formalised in the Coq proof assistant in [20], which also represents an implementation of the NBE algorithm of Section 4.

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2. Normalisation-by-evaluation as completeness

In [6], Berger and Schwichtenberg presented a proof of normalisation of $\lambda^{\to}$ which does not involve reasoning about the associated reduction relation. Instead, they interpreted λ-terms in a domain, or ambient meta-language, using an evaluation function,

$$[\![-]\!] : \Lambda \to D,$$

and then they defined an inverse to this function, which from the denotation in D directly extracts a term in $\beta\eta$-long normal form. The inverse function $\downarrow$, called *reification*, is defined by recursion on the type τ of the term, at the same time with an auxiliary function $\uparrow$, called *reflection*:

$$\downarrow^{\tau} : D \to \Lambda\text{-nf}$$
$$\downarrow^{\tau} := a \mapsto a \quad \tau\text{-atomic}$$
$$\downarrow^{\tau\to\sigma} := S \mapsto \lambda a. \downarrow^{\sigma}\big(S \cdot \uparrow^{\tau} a\big) \quad a\text{-fresh}$$

$$\uparrow^{\tau} : \Lambda\text{-ne} \to D$$
$$\uparrow^{\tau} := a \mapsto a \quad \tau\text{-atomic}$$
$$\uparrow^{\tau\to\sigma} := e \mapsto S \mapsto \uparrow^{\sigma} e\big(\downarrow^{\tau} S\big)$$

Here, S ranges over members of D, and we used $\mapsto$ and $\cdot$ for abstraction and application at the meta-level. The classes of neutral (Λ-ne) and λ-terms in normal form (Λ-nf) are given by the following inductive definition.[2]

$$\Lambda\text{-nf} \ni r := \lambda a^{\tau}.r^{\sigma} \mid e^{\tau} \quad \lambda\text{-terms in normal form}$$
$$\Lambda\text{-ne} \ni e := a^{\tau} \mid e^{\tau\to\sigma} r^{\tau} \quad \text{neutral } \lambda\text{-terms}$$

It was a subsequent realisation of Catarina Coquand [8], that the evaluation algorithm $[\![\cdot]\!]$ is also the one underlying the Soundness Theorem for minimal intuitionistic logic (with $\Rightarrow$ as the sole logical connective) with respect to Kripke models, and that the reification algorithm $\downarrow$ is also the one underlying the corresponding Completeness Theorem.

More precisely, the following well-known statements hold and their proofs have been machine-checked [9,19] for the logic fragment generated by the connectives $\{\Rightarrow, \wedge, \forall\}$.

Definition 2.1. A *Kripke model* is given by a preorder $(K, \leqslant)$ of *possible worlds*, a *quantification domain* $D(w)$ for every $w \in K$, and a relation of *forcing*, $w \Vdash X$, that interprets the predicate $X(x_1, \ldots, x_n)$ in the world w by an n-ary relation on $D(w)$, such that,

$$\text{for all } w' \geqslant w, \quad D(w) \subseteq D\big(w'\big), \quad \text{and}$$
$$\text{for } w' \geqslant w,\ d_1, \ldots, d_n \in D(w), \quad (w \Vdash X)(d_1, \ldots, d_n) \to \big(w' \Vdash X\big)(d_1, \ldots, d_n).$$

The relation of forcing is then extended to all formulae by the following clauses, using an explicit superscript σ substitution necessary for a precise handling of quantifiers:

$$w \Vdash^{\sigma} X(x_1, \ldots, x_n) := (w \Vdash X)(d_1, \ldots, d_n), \quad \text{when } \sigma = \{x_1 \mapsto d_1, \ldots, x_n \mapsto d_n\}$$
$$w \Vdash^{\sigma} A \wedge B := w \Vdash^{\sigma} A \text{ and } w \Vdash^{\sigma} B$$
$$w \Vdash^{\sigma} A \vee B := w \Vdash^{\sigma} A \text{ or } w \Vdash^{\sigma} B$$
$$w \Vdash^{\sigma} A \Rightarrow B := \text{for all } w' \geqslant w,\ w' \Vdash^{\sigma} A \to w' \Vdash^{\sigma} B$$
$$w \Vdash^{\sigma} \forall x.A(x) := \text{for all } w' \geqslant w \text{ and } d \in D\big(w'\big),\ w' \Vdash^{\sigma, x\mapsto d} A(x)$$
$$w \Vdash^{\sigma} \exists x.A(x) := \text{for some } d \in D(w),\ w \Vdash^{\sigma, x\mapsto d} A(x)$$
$$w \Vdash^{\sigma} \bot := \text{false}$$
$$w \Vdash^{\sigma} \top := \text{true}$$

We write $w \Vdash^{\sigma} \Gamma$ when w forces each formula of Γ. We write $\sigma \in D(w)$ to emphasise that all interpretations of individuals from σ live in $D(w)$.

[2] Neutral terms are the subset of normal terms that cannot be reduced on their own, whose reduction is blocked because of a free-variable appearance. Closed $\lambda^{\to}$-terms always reduce to closed terms in normal form, never to neutral terms.