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Applicative theories for the polynomial hierarchy of time and its levels ☆

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ABSTRACT

In this paper we introduce applicative theories which characterize the polynomial hierarchy of time and its levels. These theories are based on a characterization of the functions in the polynomial hierarchy using monotonicity constraints, introduced by Ben-Amram, Loff, and Oitavem.

1. Introduction

Considering theories which characterize classes of computational complexity, there are three different approaches: in one, the functions which can be defined within the theory are "automatically" within a certain complexity class. In such an account, the syntax has to be restricted to guarantee that one stays in the appropriate class. This results, in general, in the problem that certain definitions of functions do not work any longer, even if the function is in the complexity class under consideration. In a second account, the underlying logic is restricted as, for instance, in linear logic.[1] In the third account, one does neither restrict the syntax nor the logic. One allows, in general, to write down "function terms" for arbitrary (partial recursive) functions. Only for those function terms which belong to the complexity class under consideration one can *prove* that they have a certain characteristic property: usually, the property that they are "provably total" (see Definition 17 below).

Here, we follow the third account, using *applicative theories* as underlying framework.

Applicative theories are the first-order part of Feferman's system of explicit mathematics [9,10]. They provide a very handy framework to formalize theories of different strength, including to characterize classes of computational complexity.

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[1] As another example we may cite [19].

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A first characterization of polynomial time operations in applicative theories was given by Strahm in [23]. A uniform approach to various complexity classes, including FPTIME, FPSPACE, FPTIME-FLINSPACE, and FLINSPACE was given by the same author in his Habilitationsschrift, published in [24] (see also [26]). These characterizations are based on bounded schemes in the vein of Cobham [8] (see also [7]). Cantini [5] also gave a characterization of FPTIME in an applicative framework, but following the approach of Bellantoni and Cook [2] which separates the input positions of functions in normal and safe. This approach was extended by Calamai and Cantini [4,6] (see also [26, §7.2]) to a characterization of FPSPACE using a tiered characterization of FPSPACE with pointers of Oitavem [17]. Other characterizations of classes of computational complexity in applicative theories and explicit mathematics can be found, for instance, in [20–22], or [25].

On the base of a characterization of the functions in the Polynomial Hierarchy of time (FPH) which uses a monotonicity constraint given in [3], we present here an applicative theory for FPH and its levels. Given a function algebra, the main challenge concerning the design of a corresponding theory is, of course, to introduce an appropriate induction scheme which allows to prove properties for the functions under consideration. In Section 2 we rewrite the input-sorted characterization of FPH given in [3] as a non-sorted characterization in Cobham style, by introducing bounds in the recursion schemes. The next sections are concerned with the main goal of this paper: to define an induction scheme which takes care of the monotonicity constraint (see also Appendix A). While the proof of the lower bound follows from a (more or less) straightforward embedding of the function algebra described in Section 2, the upper bound is carried out by an adaptation of the proof(s) given by Strahm in [24].

Note, that Strahm also treats the polynomial hierarchy in [24], but in a quite different way which involves a special type two functional.

2. Function algebras for FPH and its levels

In this section we recap a function algebra for FPH with bounded recursion schemes in the vein of Cobham's characterization of FPtime combined with a monotonicity constraint. The underlying idea is based on previous work of the second author with Ben-Amram and Loff [3], where an input-sorted function algebra for FPH is described. For details of the function algebra presented here and its relation to the input-sorted version see [14].

Recall that PH, the polynomial hierarchy of time, is defined as $\bigcup_i \Sigma_i$ or $\bigcup_i \Delta_i$ with $\Sigma_0 = \Delta_0 = \mathrm{P}$ and, for $i \geqslant 0$, $\Sigma_{i+1} = \mathrm{NP}(\Sigma_i)$ and $\Delta_{i+1} = \mathrm{P}(\Sigma_i)$. The corresponding function classes are $\square_i = \mathrm{FPTIME}(\Delta_i) = \mathrm{FPTIME}(\Sigma_{i-1})$, for $i \geqslant 1$, and $\mathrm{FPH} = \bigcup_i \square_i = \mathrm{FPTIME}(\mathrm{PH})$.

Notation. We use $\mathbb{W}$ to denote the word algebra generated by ϵ (source), and S_0 and S_1 (successors). $\mathbb{W}$ is interpreted over the set of binary words $\{0,1\}^*$. Given $x, y \in \mathbb{W}$, $|x|$ is the length of x and $x|_y$ denotes the word corresponding to the first $|y|$ bits of x. x' denotes the numeric successor of x, and is defined by the equations $\epsilon' = S_0(\epsilon)$, $(S_0(x))' = S_1(x)$ and $(S_1(x))' = S_0(x')$. For $i \leqslant |x|$, $\mathrm{bit}_i(x)$ denotes the ith bit of x. We write $x \preccurlyeq y$ if $|x| < |y|$, or $|x| = |y|$ and $\forall i.\mathrm{bit}_i(x) \leqslant \mathrm{bit}_i(y)$. We write $x \prec y$ if $x \preccurlyeq y$ but $x \neq y$. Usually, the letters $x, y, z, w, \ldots$ denote variables, while $f, g, h, s, r, \ldots$ denote function symbols. $\vec{x}$ and $\vec{f}$ denote, respectively, a sequence of variables and functions of the appropriate arity.

Definition 1. A function h is called *inflationary* if, for all $\vec{x}, z \in \mathbb{W}$, $z \preccurlyeq h(\vec{x}, z)$.

Definition 2. Given a function h, its *inflationary section* is the function

$$h^{\uparrow}(\vec{x}, z) = \begin{cases} h(\vec{x}, z) & \text{if } z \preccurlyeq h(\vec{x}, z), \\ z & \text{otherwise.} \end{cases}$$

Clearly, inflationary sections are always inflationary functions.

Consider $[\mathcal{I}; C, BRN, BPR]$, the inductive closure of $\mathcal{I}$ under C, BRN, and BPR, where:

- $\mathcal{I}$ is the set of initial functions:
  1. ϵ,
  2. $S_i(x) = xi$, $i \in \{0,1\}$,
  3. $\pi^n_j(x_1, \ldots, x_n) = x_j$, $1 \leqslant j \leqslant n$,
  4. $Q(\epsilon, y, z_0, z_1) = y$, $Q(xi, y, z_0, z_1) = z_i$, $i \in \{0,1\}$,
  5. $\times(x, y) = 1^{|x| \times |y|}$.
- C, BRN and BPR are the following operators:
  - Composition: Given g and $\vec{h}$, their composition $f = C(g, \vec{h})$ is given by $f(\vec{x}) = g(\vec{h}(\vec{x}))$.
  - Bounded recursion on notation: Given g, h_0, h_1, and t, the bounded recursion on notation $f = BRN(g, h_0, h_1, t)$ is given by

$$f(\epsilon, \vec{x}) = g(\vec{x}),$$
$$f(yi, \vec{x}) = h_i\big(y, \vec{x}, f(y, \vec{x})\big)\big|_{t(y,\vec{x})}, \quad i \in \{0,1\}.$$