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# Epsilon substitution for first- and second-order predicate logic

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This paper is dedicated to H. Schwichtenberg whose work and support meant so much for many years

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ABSTRACT

The epsilon substitution method was proposed by D. Hilbert as a tool for consistency proofs. A version for first order predicate logic had been described and proved to terminate in the monograph “Grundlagen der Mathematik”. As far as the author knows, there have been no attempts to extend this approach to the second order case. We discuss possible directions for and obstacles to such extensions.



## 1. Introduction

The goal of this paper is to propose a new approach to an old open problem central to Hilbert’s program. It was stated several times by Hilbert (for example [5]) and repeated by P. Bernays in his preface to [6] (translation by the present author):

“At present W. Ackermann is transforming his previous proof presented in Chapter Two of this volume to employ transfinite induction in the form used by Gentzen and cover the whole arithmetical formalism.

When this succeeds — as there are all grounds to expect — then the efficacy of the original Hilbert’s Ansatz will be vindicated. In any case already on the base of Gentzen’s proof one can be convinced that the temporary fiasco of proof theory was caused merely by too high methodological requirements imposed on that theory. Of course the decisive verdict about the fate of proof theory depends on the problem of proving the consistency of Analysis.”

The epsilon substitution method was proposed by D. Hilbert as a tool for consistency proofs. A version for first order predicate logic had been described and proved (closely following an unpublished proof by W. Ackermann) to terminate in [6]. As far as the present author knows, there have been no attempts to extend this approach to the second order case. We discuss possible directions for and obstacles to such extensions. The proof in [6] is very detailed and well motivated but the combination of these features with the additional details needed for subsystems of first-order arithmetic makes it rather difficult to read. Section 2 below provides a streamlined reformulation of the main definitions and results around the termination proof given in [6] for first order predicate logic. This allows a very easy statement in Section 3 of the proposal for a terminating epsilon substitution method for second order logic that also covers analysis, since natural numbers can

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be introduced via a well-known second order definition. Section 4 is a survey of some obstacles to a termination proof and suggestions for easier intermediate results on the way to this goal.

We use $\equiv$ for syntactic coincidence of formal expressions.

## 2. First order system $\mathrm{Eps}_1$

### 2.1. $\epsilon$-Language

Our first order *$\epsilon$-language* $L^1_\epsilon$ has in addition to ordinary terms, that is variables, constants and expressions $f(t_1, \ldots, t_n)$ for functional symbols $f$, also epsilon-terms $\epsilon xF$ for arbitrary formulas $F$. Terms and formulas are defined simultaneously in the familiar way from atoms $Pt_1 \ldots t_n$ by $\&, \neg$. An $\epsilon$-term $\epsilon xF[x]$ is read "some $x$ satisfying $F[x]$".

$\epsilon x$ *binds* free occurrences of $x$ in a formula $\epsilon xF$. Substitution $F[x/t]$ of all free occurrences of a variable $x$ in $F$ by a term $t$ is defined in the standard way with the stipulation that no occurrence of a free variable of $t$ becomes bound as a result, or some bound variables are renamed to avoid that. When $x$ is clear from the context we write $F[t]$ for $F[x/t]$.

Quantifiers can be defined by

$$\exists xF[x] \leftrightarrow F\big[\epsilon xF[x]\big],$$
$$\forall xF[x] \leftrightarrow F\big[\epsilon x\neg F[x]\big]. \tag{1}$$

*Axioms of* $\mathrm{Eps}_1$ are all tautologies and *critical formulas*

$$F[t] \to F\big[\epsilon xF[x]\big]. \tag{2}$$

$\epsilon xF[x]$ is the *principal term* of this critical formula.

The only rule is *Modus Ponens*.

Definition (1) determines the *translation* of formulas of first order logic using existential quantifiers into formulas of $L^1_\epsilon$.

$$\big(\exists xF[x]\big)^\epsilon := F^\epsilon\big[\epsilon xF^\epsilon[x]\big],$$
$$(F\&G)^\epsilon := \big(F^\epsilon\&G^\epsilon\big), \qquad (\neg F)^\epsilon := \neg\big(F^\epsilon\big),$$
$$A^e := A \quad \text{for atomic formulas } A.$$

The principal part of a proof (derivation) in $\mathrm{Eps}_1$ is

a finite set $E$ of critical formulas.

$\&E$ will denote the conjunction of the (critical) formulas in $E$.

**Lemma 1.** $\mathrm{Eps}_1 \vdash G$ *iff there is a finite set* $E$ *of critical formulas such that*

$$\&E \to G \tag{3}$$

*is a tautology.*

**Proof.** Obvious induction on derivations. □

### 2.2. The substitution method from [6]

The goal (explained in [6]): find for each finite set $E$ of critical formulas a series of substitutions of $\epsilon$-terms by other terms

$$S \equiv (e_1, t_1), \ldots, (e_k, t_k)$$

turning some disjunction of instances of $E$ into a tautology. The substitutions are performed sequentially (cf. (13)).

Such a series of substitutions is a *solution for E*.

**Example 1.** $\exists xM[x]$ with quantifier-free $M[x]$.

If $\exists xM[x]$ is derivable in ordinary first order logic then for some finite set $E$ of critical formulas

$$\&E \to M\big[\epsilon xM[x]\big]$$

is a tautology. A solution for $E$ provides a tautological disjunction

$$M[t_1] \vee \cdots \vee M[t_k]$$

that is a Herbrand disjunction.

This was a nucleus of the first consistency proofs for subsystems of arithmetic with restricted induction.