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Minimal from classical proofs

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ABSTRACT

Let A be a formula without implications, and Γ consist of formulas containing disjunction and falsity only negatively and implication only positively. Orevkov (1968) and Nadathur (2000) proved that classical derivability of A from Γ implies intuitionistic derivability, by a transformation of derivations in sequent calculi. We give a new proof of this result (for minimal rather than intuitionistic logic), where the input data are natural deduction proofs in long normal form (given as proof terms via the Curry–Howard correspondence) involving stability axioms for relations; the proof gives a quadratic algorithm to remove the stability axioms. This can be of interest for computational uses of classical proofs.

It is well-known that in certain situations classical provability implies constructive provability. Glivenko proved in [5] that every negated propositional formula provable in classical logic is provable intuitionistically. Another famous "Glivenko-style" result is Barr's theorem [1], which deals with geometric formulas $\exists_{\vec{x}}(B_1 \vee \cdots \vee B_n)$ (B_i conjunctions of prime formulas) and geometric implications $\forall_{\vec{x}}(B \to \exists_{\vec{y}}(B_1 \vee \cdots \vee B_k))$ (B, B_i conjunctions of prime formulas). Barr's theorem says that for Γ consisting of geometric implications and A a geometric formula, classical derivability of A from Γ implies intuitionistic derivability. A systematic study of such theorems has been undertaken by Orevkov [11] (cf. [7] for a good survey, [8] for very clear proofs and [8,9,12] for related work). We consider one Glivenko-style theorem of Orevkov, where the conclusion is $\to$-free, and the premises contain $\to$ only positively and $\vee$, $\perp$ only negatively. We give a new proof of this result (for minimal rather than intuitionistic logic), which is of interest when computational uses of classical proofs are envisaged, as in [2]. Clearly model-theoretic arguments do not help here; one needs proof transformations. But even that is not always good enough: the way proofs are represented as input data matters. In [8,9,11] proofs are given as derivations in a sequent calculus. However, for a computational analysis natural deduction proofs are more appropriate, since by the Curry–Howard correspondence they can directly be viewed as λ-terms. A proof of Orevkov's theorem in this setting then amounts to an analysis of possible occurrences of stability axioms, and a method to eliminate them. This is what will be done in the present paper.

In Section 1 we fix our terminology for natural deduction proofs in minimal logic, and describe the standard embedding of classical logic into its $\to, \forall, \wedge$-fragment. To prepare for the proof of the main result in Section 3, we recall in Section 2 the relevant notions, as far as they are necessary to follow the proof. Section 4 discusses the algorithm provided by the proof, and the final Section 5 gives an application.[1]

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[1] We are grateful to Thierry Coquand for bringing this example to our attention.

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Table 1
Derivation terms for $\to$ and $\forall$.

Derivation	Term
$u\colon A$	u^A
$\dfrac{\begin{matrix}[u\colon A]\\ \mid M\\ B\end{matrix}}{A\to B}\to^+ u$	$(\lambda_{u^A} M^B)^{A\to B}$
$\dfrac{\begin{matrix}\mid M\\ A\to B\end{matrix}\qquad\begin{matrix}\mid N\\ A\end{matrix}}{B}\to^-$	$(M^{A\to B}N^A)^B$
$\dfrac{\begin{matrix}\mid M\\ A\end{matrix}}{\forall_x A}\forall^+ x$ (with var. cond.)	$(\lambda_x M^A)^{\forall_x A}$ (with var. cond.)
$\dfrac{\begin{matrix}\mid M\\ \forall_x A(x)\end{matrix}\qquad r}{A(r)}\forall^-$	$(M^{\forall_x A(x)}r)^{A(r)}$

Table 2
Derivation terms for $\vee$, $\wedge$ and $\exists$.

Derivation	Term
$\dfrac{\begin{matrix}\mid M\\ A\end{matrix}}{A\vee B}\vee_0^+ \qquad \dfrac{\begin{matrix}\mid M\\ B\end{matrix}}{A\vee B}\vee_1^+$	$(\vee_{0,B}^+ M^A)^{A\vee B}\ (\vee_{1,A}^+ M^B)^{A\vee B}$
$\dfrac{\begin{matrix}\mid M\\ A\vee B\end{matrix}\qquad\begin{matrix}[u\colon A]\\ \mid N\\ C\end{matrix}\qquad\begin{matrix}[v\colon B]\\ \mid K\\ C\end{matrix}}{C}\vee^- u, v$	$(M^{A\vee B}(u^A.N^C, v^B.K^C))^C$
$\dfrac{\begin{matrix}\mid M\\ A\end{matrix}\qquad\begin{matrix}\mid N\\ B\end{matrix}}{A\wedge B}\wedge^+$	$\langle M^A, N^B\rangle^{A\wedge B}$
$\dfrac{\begin{matrix}\mid M\\ A\wedge B\end{matrix}\qquad\begin{matrix}[u\colon A]\quad[v\colon B]\\ \mid N\\ C\end{matrix}}{C}\wedge^- u, v$	$(M^{A\wedge B}(u^A, v^B.N^C))^C$
$\dfrac{r\qquad\begin{matrix}\mid M\\ A(r)\end{matrix}}{\exists_x A(x)}\exists^+$	$(\exists_{x,A}^+ rM^{A(r)})^{\exists_x A(x)}$
$\dfrac{\begin{matrix}\mid M\\ \exists_x A\end{matrix}\qquad\begin{matrix}[u\colon A]\\ \mid N\\ B\end{matrix}}{B}\exists^- x, u$ (var. cond.)	$(M^{\exists_x A}(x, u^A.N^B))^C$ (var. cond.)

1. Minimal logic

Natural deduction is a distinguished logical system, since it allows to formalize faithfully proofs done by a mathematician who wants to write out all details; this was convincingly spelled out by Gentzen [4]. On the more technical side, natural deduction corresponds closely to the simply typed λ-calculus ("Curry–Howard correspondence"). This is particularly so if we define negation by $\neg A := A \to \bot$ with $\bot$ just a distinguished propositional symbol; the resulting system is *minimal logic*. We then can add extra axioms for $\bot$ (e.g., stability or ex-falso-quodlibet) to embed classical or intuitionistic logic. In minimal logic, for each of the connectives $\to$, $\forall$ and also $\exists$, $\vee$ and $\wedge$ we have introduction and elimination rules (I-rules and E-rules) given in Tables 1 and 2. The left premise $A \to B$ in $\to^-$ is called the *major* (or *main*) premise, and the right premise A the *minor* (or *side*) premise. Similarly, in each of the elimination rules $\vee^-$, $\wedge^-$ and $\exists^-$ the left premise is called *major* (or *main*) premise, and the right premise is called the *minor* (or *side*) premise. We define the *weak* variants $\tilde{\exists}, \tilde{\vee}$ of $\exists, \vee$ by

$$\tilde{\exists}_x A := \neg\forall_x\neg A \quad \text{and} \quad A \mathbin{\tilde{\vee}} B := \neg A \to \neg B \to \bot.$$

Clearly $\vdash \exists_x A \to \tilde{\exists}_x A$ and $\vdash A \vee B \to A \mathbin{\tilde{\vee}} B$, but not conversely; this is the reason why $\tilde{\exists}, \tilde{\vee}$ are called "weak".

The *stability* axioms are of the form $\forall_{\vec{x}}(\neg\neg P\vec{x} \to P\vec{x})$ with P a relation symbol distinct from $\bot$. It is easy to see that from the stability axioms we can derive $\neg\neg A \to A$ for every formula A built with $\to$, $\forall$, $\wedge$ only. Let Stab denote the set of all stability axioms. We write $\Gamma \vdash_c B$ for $\Gamma \cup \text{Stab} \vdash B$, and call B *classically derivable* from Γ. Similarly, let Efq denote