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Aligning the weak König lemma, the uniform continuity theorem, and Brouwer's fan theorem

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ABSTRACT

We characterize the uniform continuity theorem for integer-valued functions on Cantor space and Brouwer's fan theorem by expressing them as variations of the weak König lemma.

The weak König lemma (WKL), which says that *every infinite binary tree has an infinite path*, plays a role in various fields of mathematics. For instance, it represents a hierarchy in classical reverse mathematics [10], where it is considered a set-existence axiom. In constructive formal systems like HA^ω, it can be written as the combination of an omniscience principle and a choice axiom [7].

The system HA^ω is based on finite types and recursive functions. Neither the law of excluded middle nor the axiom of choice is presupposed. The system EL has similar features, but deals with numbers and functions from numbers to numbers only [11]. We write BISH for constructive mathematics in the style of Bishop [3]. This is informal mathematics on the base of intuitionistic logic. One difference between BISH and, say HA^ω, lies in the status of choice axioms. They are largely permitted in the former but not a part of the latter.

The uniform continuity theorem (UC) says that every pointwise continuous function $F : \{0, 1\}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous.

Brouwer's fan theorem (FAN) for detachable bars, which says that *every bar is uniform*, is the contrapositive of WKL and therefore, with classical logic, equivalent to WKL. However, from a constructive point of view it is much weaker, since it is an essential part of Brouwer's intuitionism [6].

We know that:

- WKL implies FAN in EL [8]
- WKL implies UC in HA^ω [2]
- UC implies FAN in BISH (easy to see)
- UC does not imply WKL in BISH (as long as Brouwer's intuitionism is not inconsistent) [5]
- FAN does not hold in BISH (as long as constructive recursive mathematics is not inconsistent) [5]

These are open questions:

[A] does UC imply FAN in HA^ω?

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[B] does FAN imply UC in HA^{ω}?
[C] does FAN imply UC in BISH?

The obvious proof of the implication UC $\Rightarrow$ FAN in BISH depends heavily on choice, therefore it is of little help for solving problem [A]. If we assume that pointwise continuous functions come with a modulus, the answer to [C] is 'yes' [1]. However, this constitutes a loss of generality in the concept of continuity [9]. The restricted concept is often accepted when it comes to coding functions, which explains the widespread belief that UC and FAN are equivalent. In its general form, the question [C] remains open.

Working within BISH, we formulate both UC and FAN as instances of WKL. This yields a new proof of

$$\text{WKL} \rightarrow \text{UC} \rightarrow \text{FAN}.$$

Furthermore, with this representation, UC is structurally more complex than FAN, which backs the intuition that the answer to question [C] is 'no'.

Let $\{0, 1\}^*$ denote the set of all finite binary sequences u, v, w. We write $|u|$ for the length of u and $u * v$ for the concatenation of u and v. That means, for $u = (u(0), \ldots, u(l-1))$ and $v = (v(0), \ldots, v(k-1))$, we have $|u| = l$ and

$$u * v = (u(0), \ldots, u(l-1), v(0), \ldots, v(k-1)) .$$

Let $\{0, 1\}^{\mathbb{N}}$ denote Cantor space, the set of all infinite binary sequences. We use Greek letters for the elements of $\{0, 1\}^{\mathbb{N}}$. Set

$$\overline{\alpha}n := (\alpha(0), \ldots, \alpha(n-1)) .$$

Accordingly, if $|u| \leq n$, we define

$$\overline{u}n := (u(0), \ldots, u(n-1)).$$

On $\{0, 1\}^{\mathbb{N}}$ we have the metric

$$d(\alpha, \beta) = \inf\left\{2^{-n} \mid \overline{\alpha}n = \overline{\beta}n\right\} . \tag{1}$$

By Top we denote the topology induced by this metric. For u of length n we define

$$O_u = \{\alpha \mid \overline{\alpha}n = u\} .$$

The sets of this kind form a base of the topology Top. A subset C of $\{0, 1\}^{\mathbb{N}}$ is *closed*, if it contains the limit of every convergent sequence of elements of C, where both limit and convergence are understood with respect to Top.

Let $u * \alpha$ be the concatenation of u and α, defined by

$$n \mapsto \begin{cases} u(n) & \text{if } n < |u| ; \\ \alpha(n - |u|) & \text{if } n \geq |u| . \end{cases}$$

A subset T of $\{0, 1\}^*$ is

- *detachable* if $\forall u (u \in T \vee u \notin T)$;
- *closed under restriction* if

$$\forall u, v\, (u * v \in T \rightarrow u \in T) ;$$

- an *infinite binary tree* if
  - it is detachable
  - it is closed under restriction
  - we have
$$\forall n\, \exists u\, (|u| = n \wedge u \in T) . \tag{2}$$

Let $\mathcal{T}$ denote the set of all infinite binary trees. Fix $T \in \mathcal{T}$. We define

$$\mathcal{A}_T = \{\alpha \mid \forall n\, (\overline{\alpha}n \in \mathrm{T})\} .$$

A sequence $\alpha \in \mathcal{A}_T$ is called an *infinite path* of T. The *weak König lemma* is the following axiom.

WKL Every infinite binary tree has an infinite path.

We say that $\mathcal{A}_T$ is *decidable* if

$$\forall \alpha\, (\alpha \in \mathcal{A}_T \vee \exists n\, (\overline{\alpha}n \notin T)) .$$

Note that this is stronger than the condition

$$\forall \alpha\, (\alpha \in \mathcal{A}_T \vee \alpha \notin \mathcal{A}_T) .$$

If $\mathcal{A}_T$ is decidable, we have

$$\forall \alpha\, (\alpha \notin \mathcal{A}_T \Leftrightarrow \exists n\, (\overline{\alpha}n \notin T)) .$$

Note further that $\mathcal{A}_T$ is a closed set for every $T \in \mathcal{T}$.