Annals of Pure and Applied Logic 163 (2012) 1057–1061

Contents lists available at SciVerse ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

The uniform boundedness theorem and a boundedness principle

Hajime Ishihara

School of Information Science, Japan Advanced Institute of Science and Technology, Nomi, Ishikawa 923-1292, Japan

ARTICLE INFO

Article history:
Available online 31 December 2011

MSC:
03F60
46S30

Keywords:
Constructive mathematics
Topological vector space
The uniform boundedness theorem
Boundedness principle

ABSTRACT

We deal with a form of the uniform boundedness theorem (or the Banach–Steinhaus theorem) for topological vector spaces in Bishop's constructive mathematics, and show that the form is equivalent to the boundedness principle BD-N, and hence holds not only in classical mathematics but also in intuitionistic mathematics and in constructive recursive mathematics. The result is also a result in constructive reverse mathematics.

1. Introduction

The notion of a topological vector space, as a generalization of the notion of a normed space, is a very important notion to investigate function spaces, such as the space of test functions, which do not form Banach spaces (see, for example, [23]). Nonetheless little investigation on topological vector spaces has been done in Bishop's constructive mathematics [6–8,10]; see also a discussion in [6, Appendix A]. We can only find, noting that a topological vector space is a uniform space, a constructive concept of a uniform space with a set of pseudometrics, and basic theorems, such as, that arbitrary uniform space has a completion, in [6, Problems 17 to 21 of Chapter 4]; see also [7, Problems 22 to 26 of Chapter 4], and [9,24,12,11,17,5] for other constructive treatments of a uniform space.

However, using the notion of a neighbourhood space [6, 3.3] (see also [7, 3.3]) introduced by Bishop, we can naturally define a notion of a topological vector space in Bishop's constructive mathematics as follows.

A *neighbourhood space* is a pair (X, τ) consisting of a set X and a set τ of subsets of X such that

NS1. $\forall x \in X \exists U \in \tau (x \in U)$,
NS2. $\forall x \in X \forall U, V \in \tau [x \in U \cap V \implies \exists W \in \tau (x \in W \subseteq U \cap V)]$.

The set τ is an *open base* on X, and an element of τ is a *basic open set*. A subset of X is *open* if it is a union of basic open sets. A *neighbourhood* of a point $x \in X$ is a subset $A \subseteq X$ such that $x \in U \subseteq A$ for some $U \in \tau$. An open base σ on X is *compatible* with τ if each neighbourhood in σ is a neighbourhood in τ, and vice versa. An open base is *compatible with a metric d* if it is compatible with the open base induced by open balls. A function f between neighbourhood spaces (X, τ) and (Y, σ) is *continuous* if $f^{-1}(V)$ is open for each $V \in \sigma$.

A *topological vector space* is a vector space E equipped with an open base τ such that the vector space operations, addition $(x, y) \mapsto x + y$ and scalar multiplication $(a, x) \mapsto ax$, are continuous, that is, if U is a neighbourhood of $x + y$, then there exist neighbourhoods V and V' of x and y, respectively, such that $V + V' = \{v + v' \mid v \in V, v' \in V'\} \subseteq U$, and if U is a neighbourhood of ax, then for some $\delta > 0$ and some neighbourhood V of x we have $bV = \{bv \mid v \in V\} \subseteq U$ whenever

E-mail address: ishihara@jaist.ac.jp.

0168-0072/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.apal.2011.12.027

$|a-b| < \delta$. It is *metrizable* if τ is compatible with some metric d, and is an *F-space* if its open base τ is compatible with a complete invariant metric d. Here a metric d on a vector space E is *invariant* if $d(x+z, y+z) = d(x, y)$ for all $x, y, z \in E$.

In this paper, we deal with the following form of the uniform boundedness theorem (or the Banach–Steinhaus theorem) for topological vector spaces [23, 2.6] in Bishop's constructive mathematics.

The Uniform Boundedness Theorem. *If $(T_m)_m$ is a sequence of continuous linear mappings from an F-space E into a topological vector space F such that the set*

$$\{T_m x \mid m \in \mathbf{N}\}$$

is bounded in F for each $x \in E$, then $(T_m)_m$ is equicontinuous.

Here a subset A of a topological vector space E is *bounded* if for each neighbourhood V of 0 in E there exists a positive integer K such that $A \subseteq tV$ for each $t \geq K$, and a set Γ of continuous linear mappings between topological vector spaces E and F is *equicontinuous* if for each neighbourhood V of 0 in F there exists a neighbourhood U of 0 in E such that $T(U) \subseteq V$ for each $T \in \Gamma$.

We know that a (contrapositive) form of the uniform boundedness theorem for normed spaces has a constructive proof [6, Problem 6 of Chapter 9] (see also [7, Problem 20 of Chapter 7]), and a corollary [23, Theorem 2.8] of the uniform boundedness theorem for a sequence of *sequentially continuous* linear mappings from a separable Banach space into a normed space holds constructively [14, Theorem 7]. However, the corollary for a sequence of *continuous* linear mappings not only implies, but also is equivalent to the following boundedness principle (BD-N) [15, Theorem 21].

BD-N. Every pseudobounded countable subset of $\mathbf{N}$ is bounded.

Here a subset S of $\mathbf{N}$ is *countable* if it is a range of $\mathbf{N}$, *pseudobounded* if $\lim_{n\to\infty} s_n/n = 0$ for each sequence $(s_n)_n$ in S, and *bounded* if there exists a positive integer K such that $s < K$ for each $s \in S$; see [13,18,22] for pseudobounded sets.

The boundedness principle BD-N is equivalent to the statement "every sequentially continuous mapping from a separable metric space into a metric space is continuous" [13, Theorem 4], is derivable in intuitionistic mathematics with a continuity principle [13, Proposition 3] and in constructive recursive mathematics with Church's thesis and Markov's principle [13, Proposition 4], and is not provable in $\mathbf{HA}^{\omega}$ with axiom of choice for all finite types [19].

In the following, we show that the uniform boundedness theorem for topological vector spaces with a sequence of continuous linear mappings is also equivalent to the boundedness principle BD-N, and hence holds not only in classical mathematics but also in intuitionistic mathematics and in constructive recursive mathematics. The result is also a result in constructive reverse mathematics [16,20,25].

Although the result is presented in informal Bishop-style constructive mathematics, it is possible to formalize it in constructive Zermelo–Fraenkel set theory (**CZF**), founded by Aczel [1–3], with the dependent choice axiom (DC), which permits a quite natural interpretation in Martin-Löf type theory [21]. Note that the axiom of countable choice (AC_{ω}) follows from the dependent choice axiom in **CZF**; see [4, Section 8].

2. The main results

A topological vector space E is *separated* if for each neighbourhood U of 0 there exist a neighbourhood V of 0 and an open set W such that $E = U \cup W$ and $V \cap W = \emptyset$.

Each topological vector space E whose open base is *compatible with a set $\{d_i \mid i \in I\}$ of pseudometrics*, that is, compatible with the open base consisting of the sets $B_{i_1,\ldots,i_n}(x, \epsilon) = \{y \in E \mid \sum_{k=1}^{n} d_{i_k}(x, y) < \epsilon\}$, is separated. In fact, for each neighbourhood U of 0, there exist $i_1, \ldots, i_n \in I$ and $\epsilon > 0$ such that $B_{i_1,\ldots,i_n}(0, \epsilon) \subseteq U$, and hence, taking $V = B_{i_1,\ldots,i_n}(0, \epsilon/2)$ and $W = \{y \in E \mid \epsilon/2 < \sum_{k=1}^{n} d_{i_k}(0, y)\}$, we have $E = U \cup W$ and $V \cap W = \emptyset$.

On the other hand, suppose that a vector space E with the discrete topology is separated. Then, since $\{0\}$ is a neighbourhood of 0, there exist a neighbourhood V of 0 and an open set W such that $E = \{0\} \cup W$ and $V \cap W = \emptyset$. Hence for each $x \in E$, either $x \in \{0\}$ or $x \in W$: in the former case, we have $x = 0$; in the latter case, we have $\neg(x = 0)$. If $E = \mathbf{R}$, then this is equivalent to the *weak limited principle of omniscience* (WLPO) [8, 1.1]:

$$\forall x \in \mathbf{R}[x = 0 \vee \neg(x = 0)].$$

Since it is doubtful that we can achieve a constructive proof of WLPO, we cannot find out whether the topological vector space $\mathbf{R}$ with the discrete topology is separated.

A subset A of a topological vector space E is *unbounded* if there exists a neighbourhood V of 0 in E such that for each positive integer k there exist $t \geq k$ and $x \in A$ such that $x \notin tV$.

The following theorem generalizes the constructive version of the uniform boundedness theorem [6, Problem 6 of Chapter 9] (see also [7, Problem 20 of Chapter 7]) to topological vector spaces.

Theorem 1. *Let $(T_n)_n$ be a sequence of continuous linear mappings from an F-space E into a separated topological vector space F. If there exists a bounded sequence $(x_n)_n$ in E such that $\{T_n x_n \mid n \in \mathbf{N}\}$ is unbounded, then $\{T_n x \mid n \in \mathbf{N}\}$ is unbounded for some $x \in E$.*