

Canonical structure in the universe of set theory: part two

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Abstract

We prove a number of consistency results complementary to the ZFC results from our paper [J. Cummings, M. Foreman, M. Magidor, Canonical structure in the universe of set theory: part one, *Annals of Pure and Applied Logic* 129 (1–3) (2004) 211–243]. We produce examples of non-tightly stationary mutually stationary sequences, sequences of cardinals on which every sequence of sets is mutually stationary, and mutually stationary sequences not concentrating on a fixed cofinality. We also give an alternative proof for the consistency of the existence of stationarily many non-good points, show that diagonal Prikry forcing preserves certain stationary reflection properties, and study the relationship between some simultaneous reflection principles. Finally we show that the least cardinal where square fails can be the least inaccessible, and show that weak square is incompatible in a strong sense with generic supercompactness.

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1. Introduction

In our paper [6] we prove a number of ZFC results concerning PCF theory, mutual stationarity, square principles and stationary reflection. In that paper we discussed the informal notion of *canonical structure*. This notion is supposed to capture the idea of structure that is not arbitrarily determined by non-constructive existence assumption. For example, structure that requires the axiom of choice to prove its existence may still be independent of any choices made in proving it exists. Cardinals of uncountable cofinality fall into this category. Other examples might include fine structure models of large cardinals. Large cardinal axioms are non-constructive assumptions (as opposed to e.g. the pairing axiom, where we know exactly what the intended object is). However, as a consequence of their existence there is various canonical structure, such as $U \cap L[U]$ for U a normal ultrafilter on a measurable cardinal κ .

The notion of canonical structure is different from the notion of absoluteness. We illustrate this with an example. Assuming the Axiom of Choice, the collection of real numbers has some well-ordered cardinality $\mathfrak{c}$ and this cardinality

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is independent of the choices made to show it exists. Similarly, one needs the Axiom of Choice to prove that the least regular uncountable ordinal ($\aleph_1$) exists. Both of these objects are “canonical” in our sense, but it is independent of ZFC whether they are in fact identical. We would like to say that these are distinct examples of structure that may or may not determine identical objects.

In this paper we continue to explore canonical structure by proving consistency results complementary to the ZFC results in [6].

After some preliminaries in Section 2, we show the following results.

- In Section 3, we give a forcing construction for a sequence of stationary sets which is mutually stationary but not tightly stationary. The proof involves a combinatorial principle which we dub Coherent Squares.
- In Section 4, we give another forcing construction for a sequence of stationary sets which is mutually stationary but not tightly stationary. The proof involves some lemmas about uniform structures and mutual stationarity which are of independent interest. We also show the consistency of a splitting property for mutually stationary sequences.
- In Section 5 we show that on an increasing ω -sequence of measurable cardinals, any sequence of stationary sets is mutually stationary. We also show that for any Prikry-generic sequence, a tail of the sequence has this property.
- In Section 6 we give an alternative proof of a theorem by Shelah, that there can exist sequences of stationary sets on the $\aleph_n$ for n finite which are mutually stationary and do not concentrate on a fixed cofinality.
- In Section 7 we give an alternative construction for a model in which the set of non-good points of cofinality $\aleph_1$ in $\aleph_{\omega+1}$ is non-stationary. We also show that if we are given an increasing ω -sequence of measurable cardinals such that the successor of their supremum exhibits a certain stationary reflection property, then the reflection property is preserved by diagonal Prikry forcing.
- In Section 8 we show that the principle saying that for all λ any family of fewer than η many stationary subsets of $[\lambda]^{\aleph_0}$ reflect does not imply simultaneous reflection of η many sets of ω -cofinal ordinals. The proof uses Martin’s Maximum.
- In Section 9 we show that it is consistent that the least λ for which $\square_\lambda$ fails is inaccessible.
- In Section 10 we show that if $\square_\mu^*$ holds for a singular cardinal μ of cofinality ω , then a cardinal-preserving countably closed forcing poset can not create any instances of supercompactness below μ . This shows that there is an essential problem in a result by Ben-David and Shelah [2].

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2. Preliminaries

In this section we give some background material on mutual and tight stationarity and PCF theory. For more details we refer the reader to [6].

The idea of *mutual stationarity* was introduced by Foreman and Magidor [12] in their work on the non-saturation of the non-stationary ideal on $P_\kappa \lambda$.

Definition 2.1. Let $\langle S_\kappa : \kappa \in K \rangle$ be such that $S_\kappa \subseteq \kappa$ for all $\kappa \in K$, where K is a set of regular uncountable cardinals.

- (1) If N is a set, then N *meets* $\langle S_\kappa : \kappa \in K \rangle$ if and only if $\sup(N \cap \kappa) \in S_\kappa$ for all $\kappa \in N \cap K$.
- (2) $\langle S_\kappa : \kappa \in K \rangle$ is *mutually stationary* if and only if for every algebra $\mathcal{A}$ on $\sup(K)$ there exists $N \prec \mathcal{A}$ such that N meets $\langle S_\kappa : \kappa \in K \rangle$.

If $S \subseteq P(X)$ then S is a *stationary subset* of $P(X)$ if and only if for every algebra $\mathcal{A}$ on X there is $B \in S$ such that $B \prec \mathcal{A}$.

The sequence $\langle S_\kappa : \kappa \in K \rangle$ is mutually stationary if and only if the set of subsets of $\sup(K)$ which meet $\langle S_\kappa : \kappa \in K \rangle$ is a stationary subset of $P(\sup(K))$. By standard facts [13, Lemma 0] about generalised stationarity, if X is any set with $\sup(K) \subseteq X$ then $\langle S_\kappa : \kappa \in K \rangle$ is mutually stationary if and only if the set of subsets of X which meet $\langle S_\kappa : \kappa \in K \rangle$ is a stationary subset of $P(X)$.

It is easy to see that if $\langle S_\kappa : \kappa \in K \rangle$ is mutually stationary then S_κ is stationary for each κ . Foreman and Magidor showed that the converse is false in general, but is true if $S_\kappa \subseteq \kappa \cap \text{cof}(\omega)$ for all κ . In order to get versions of Solovay’s splitting theorem and Fodor’s theorem Foreman and Magidor introduced the notion of *tight structure* and *tightly stationary sequence*.

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