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Square and non-reflection in the context of $\mathcal{P}_\kappa\lambda$

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Abstract

We define $\square_{\mathcal{P}_\kappa\lambda}$, a square principle in the context of $\mathcal{P}_\kappa\lambda$, and prove its consistency relative to ZFC by a directed-closed forcing and hence that it is consistent to have $\square_{\mathcal{P}_\kappa\lambda}$ hold when κ is supercompact, whereas $\square_\kappa$ is known to fail under this condition. The new principle is then extended to produce a principle with a non-reflection property. Another variation on $\square_{\mathcal{P}_\kappa\lambda}$ is also considered, this one based on a family of club subsets of $\mathcal{P}_{\kappa_x}(x)$. Finally, a new square principle for cardinals, denoted $\square^f_\kappa$, is introduced. This principle is proved consistent with κ being supercompact. It is shown to yield a non-reflection result similar to that given by $\square_\kappa$.

Keywords: $\mathcal{P}_\kappa\lambda$; Square; Non-reflection

1. Introduction

The principal aim of this paper is to generalise the square principle to the context of $\mathcal{P}_\kappa\lambda$, where for infinite cardinals $\kappa \leq \lambda$, $\mathcal{P}_\kappa\lambda$ is the set of subsets of λ of size $<\kappa$. The combinatorial research that we present follows a well-established tradition and is principally guided by the idea of transferring useful notions from the theory of the combinatorics of ordinal numbers. While Jensen's diamond principle (see [8]) has been usefully generalised to this context (originally by Jech in [5], but also by Matet in [13] and [12] and by Džamonja in [3]), the square principle had not been prior to the work presented here.

We begin by establishing the main properties of the $\square_\kappa$ principle. We then use $\square_\kappa$ to construct a $\square_{\mathcal{P}_\kappa\lambda}$ principle. We then prove its relative consistency to ZFC by forcing. A major point of interest here is that we can have $\square_{\mathcal{P}_\kappa\lambda}$ hold when κ is supercompact, whereas $\square_\kappa$ is known to fail under this condition. The $\square_{\mathcal{P}_\kappa\lambda}$ principle serves as a foundation on which further properties, in particular non-reflection, can be added. Another variation on the $\square_{\mathcal{P}_\kappa\lambda}$ principle is considered, this one based on a family of club subsets of $\mathcal{P}_{\kappa_x}(x)$. We close by using $\square_{\mathcal{P}_\kappa\lambda}$ to inspire a new square principle in the context of the ordinals. This principle, denoted $\square^f_\kappa$, is consistent with κ being supercompact and yields a non-reflection result comparable to that given by $\square_\kappa$.

Throughout this paper, κ is a regular infinite cardinal and λ is an infinite cardinal with $\kappa \leq \lambda$. We now define $\mathcal{P}_\kappa\lambda$. Note that κ and λ are arguments and may be replaced by specified cardinals or sets respectively. For example, in this

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paper we will often consider $\mathcal{P}_{|x|}(x)$ where x is a set. Note that $\mathcal{P}_\kappa\lambda$ is also commonly written as $[\lambda]^{<\kappa}$ although this notation is avoided in this paper.

Definition 1.1. Let $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda : |x| < \kappa\}$. More generally, if y is an arbitrary set, then $\mathcal{P}_\kappa(y) = \{x \subseteq y : |x| < \kappa\}$.

The notation used throughout this paper is standard in set theory, but some explanations are necessary for some basic symbols. By $x \subset y$ we mean $x \subseteq y$ and $x \neq y$. Where we deal specifically with elements and subsets of $\mathcal{P}_\kappa\lambda$ we typically use lower case Roman letters $x, y, \ldots$ for elements and upper case Roman letters $X, Y, \ldots$ for subsets. We use script uppercase letters, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ for sets of subsets of $\mathcal{P}_\kappa\lambda$. We use the lower case Greek alphabet $\alpha, \beta, \ldots$ for ordinals and letters from κ onwards for cardinals. We write $\lim(\alpha)$ as an abbreviation for "α is a limit ordinal". We use $\kappa^{(+n)}$ to mean the nth cardinal after κ. When n is 1 or 2 we write κ^+ or κ^{++} respectively. We denote the order type of an ordinal α by $\mathrm{otp}(\alpha)$. For any function, say f, we write $\mathrm{dom}(f)$ and $\mathrm{im}(f)$ respectively for the domain and image of f. If f is a function and X is a set then $f\lceil X = \{(x, f(x)) : x \in X\}$. That is, $f\lceil X$ is the function given by restricting the domain of f to X. In forcing proofs, we will follow the convention that for conditions p, q of a forcing notion $(PO, \leq)$, $p \leq q$ implies that p is a weaker condition than q, in the sense that it offers less information. We will also use $\geq$, $<$ and $>$ in the natural way.

The square principle, denoted $\square_\kappa$, was developed by Jensen and has proved a useful tool in various areas of mathematical logic.

Definition 1.2. $\square_\kappa$ is the statement that there is a sequence $\langle C_\alpha : \alpha \in \kappa^+, \lim(\alpha)\rangle$ with the following properties:

(i) C_α is a club subset of α.
(ii) If $\mathrm{cf}(\alpha) < \kappa$ then $\mathrm{otp}(C_\alpha) < \kappa$.
(iii) (Coherence:) If $\beta \in C_\alpha$ and $\lim(\beta)$ then $C_\beta = C_\alpha \cap \beta$.

Forcing can be used to produce a model of set theory in which $\square_\kappa$ holds. This approach uses a partial order whose elements are initial segments of potential square sequences. It is also known that $\square_\kappa$ holds in L, the universe of constructible sets. The proof uses fine structure theory and is due to Jensen and given in [8]; alternative accounts can be found in [2] and [4].

The $\square_\kappa$ principle defined above encapsulates two distinct properties, namely non-reflection and coherence, both of which have proved fruitful in combinatorial research.

The non-reflection theorem based upon $\square_\kappa$ makes use of Fodor's Lemma. This well-known lemma can be found in most set theory textbooks, for example in [6]. We present it here without proof.

Lemma 1.3 (*Fodor*). *Suppose that S is a stationary subset of a regular cardinal μ. Suppose also that $f : S \to \mu$ is such that $f(\alpha) < \alpha$ for all $\alpha \in S$. Then there is a stationary subset $T \subseteq S$ such that f is constant on T.*

Theorem 1.4. *If $\square_\kappa$ holds then κ^+ has a non-reflecting stationary subset.*

Proof. Suppose $\langle C_\alpha : \alpha < \kappa^+ \text{ and } \lim(\alpha)\rangle$ is as specified in the definition of $\square_\kappa$. Let $T = \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) < \kappa < \alpha$ and $\lim(\alpha)\}$. To see that this is stationary, let C be an arbitrary club of κ^+ and let C^* be the club given by $C\backslash\kappa$. Then the ωth element of C^* is an element of T.

Now define $F : T \to \kappa$ by $F(\alpha) = \mathrm{otp}(C_\alpha)$. By part (ii) of Definition 1.2 and the definition of T, $F(\alpha) < \kappa < \mathrm{otp}(\alpha)$ for all $\alpha \in T$. Hence, by Lemma 1.3, we can select a stationary subset $R \subseteq T$ such that F is constant on R.

Now suppose R reflects in α for some $\alpha \in R$. Let $\beta, \gamma \in R \cap C_\alpha$ with $\beta < \gamma$. Then $C_\beta \cup \{\beta\} \subseteq C_\gamma$ as $\beta = \sup(C_\beta)$. Thus $F(\gamma) = \mathrm{otp}(C_\gamma) \geq \mathrm{otp}(C_\beta) + 1 > F(\beta)$. But this is a contradiction because F is constant on R. $\dashv$

The proof given above introduces regressive function given by $f(\alpha) = \mathrm{otp}(C_\alpha)$ in order to invoke Fodor's Lemma. The proofs of Theorems 3.2 and 5.16 follow a similar strategy but in these cases, the regressive functions are given explicitly by the relevant principles.

2. A square principle for $\mathcal{P}_\kappa\lambda$

After an initial brief study of a trivial square-like principle, we define the $\square_{\mathcal{P}_\kappa\lambda}$ principle and use forcing to establish its consistency relative to ZFC. Importantly, in this forcing we can preserve the supercompactness of cardinals $\leq\kappa$