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# Coding by club-sequences

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**Abstract**

Given any subset $A$ of $\omega_1$ there is a proper partial order which forces that the predicate $x \in A$ and the predicate $x \in \omega_1 \setminus A$ can be expressed by $ZFC$-provably incompatible $\Sigma_3$ formulas over the structure $\langle H_{\omega_2}, \in, NS_{\omega_1} \rangle$. Also, if there is an inaccessible cardinal, then there is a proper partial order which forces the existence of a well-order of $H_{\omega_2}$ definable over $\langle H_{\omega_2}, \in, NS_{\omega_1} \rangle$ by a provably antisymmetric $\Sigma_3$ formula with two free variables. The proofs of these results involve a technique for manipulating the guessing properties of club-sequences defined on stationary subsets of $\omega_1$ at will in such a way that the $\Sigma_3$ theory of $\langle H_{\omega_2}, \in, NS_{\omega_1} \rangle$ with countable ordinals as parameters is forced to code a prescribed subset of $\omega_1$. On the other hand, using theorems due to Woodin it can be shown that, in the presence of sufficiently strong large cardinals, the above results are close to optimal from the point of view of the Levy hierarchy.


*Keywords:* Coding into $H_{\omega_2}$; Optimal definitions over $\langle H_{\omega_2}, \in, NS_{\omega_1} \rangle$; Club-guessing properties of club-sequences; Specifiable sequences of club-sequences

## 1. Introduction and background

The present article deals with the problem of finding optimal definitions of well-orders of the reals and other objects. More precisely, it addresses the following two questions:

**Question 1.** Suppose $A$ is a subset of $\omega_1$. Suppose we are given the task of going over to a *nice* set-forcing extension[1] in which $A$ admits a *simple* definition $\Phi(x)$, without parameters, over the structure $\langle H_{\omega_2}, \in \rangle$ (or over some natural extension of this structure, like $\langle H_{\omega_2}, \in, NS_{\omega_1} \rangle$). What is the lowest degree of logical complexity that can be attributed to a definition $\Phi(x)$ for which we can perform the above task?

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[1] So, if *nice* is to be interpreted as preserving stationary subsets of $\omega_1$ in the ground model – which indeed is the interpretation that we shall adopt – and $A$ is a stationary and co-stationary subset of $\omega_1$, then $A$ will remain stationary and co-stationary in the extension.

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**Question 2.** What is the lowest possible degree of logical complexity of formulas for which there is a formula $\Phi(x, y)$ (again without parameters) with that complexity and with the property that we can go over to a set-forcing extension in which the set of real numbers admits a well-order defined by $\Phi(x, y)$ (again over the structure $\langle H_{\omega_2}, \in\rangle$ or over some natural extension of it)?

We will measure logical complexity by means of the familiar Levy hierarchy $\bigcup_{n<\omega}\{\Sigma_n, \Pi_n\}$ of formulas. Recall that the $\Sigma_n$ and $\Pi_n$ formulas of a language extending the language of set theory are defined by specifying that a formula is $\Sigma_0$ (equivalently, $\Pi_0$) if all of its quantifiers are restricted[2] and by specifying, for $n > 0$, that a formula is $\Sigma_n$ (respectively, $\Pi_n$) if it is of the form $(\exists x)\varphi$ for a $\Pi_{n-1}$ formula $\varphi$ (respectively, if it is of the form $(\forall x)\varphi$ for a $\Sigma_{n-1}$ formula $\varphi$). We may also say that a formula $\varphi(x_0, \ldots, x_k)$ is $\Delta_n$ if there is a $\Sigma_n$ formula $\varphi_0$ and a $\Pi_n$ formula $\varphi_1$ such that $\varphi$ is logically equivalent to both $\varphi_0$ and $\varphi_1$ (that is, if $(\forall x_0, \ldots, x_k)[(\varphi(x_0, \ldots, x_k) \leftrightarrow \varphi_0(x_0, \ldots, x_k)) \wedge (\varphi(x_0, \ldots, x_k) \leftrightarrow \varphi_1(x_0, \ldots, x_k))]$ holds). It is clear that $\Sigma_n \cup \Pi_n \subseteq \Delta_{n+1} \subseteq \Sigma_{n+1} \cap \Pi_{n+1}$ holds for every $n < \omega$. Also, note that, in any model $M$ of *ZF* without the Power Set Axiom, if $P$ is a definable class in $M$ and $\varphi(x_0, \ldots, x_k)$ is a formula of the language of the structure $\langle M, \in, P\rangle$, then there is some formula $\psi(x_0, \ldots, x_k) \in \bigcup_{n<\omega}\{\Sigma_n, \Pi_n\}$ (of the same language) such that, in $\langle M, \in, P\rangle$, $\varphi(x_0, \ldots, x_k)$ is logically equivalent to $\psi(x_0, \ldots, x_k)$. In other words, the Levy hierarchy provides a classification, up to logical equivalence, of all formulas over such structures $\langle M, \in, P\rangle$.

We shall be addressing the above questions under the assumption that generously powerful large cardinals are at hand. More precisely, we shall perform the tasks expressed in Questions 1 and 2, namely the construction of the forcing extensions in which there are simple definitions of the relevant objects, under no extra assumption – for Question 1 – and under the assumption that there is an inaccessible cardinal – for Question 2 –, and afterwards we will prove, under the extra assumption that there is a proper class of Woodin cardinals or a proper class of measurable Woodin cardinals, that there are certain limitations to carrying out the above tasks in a more efficient way (from the point of view of the logical complexity of the definitions).

*ZFC*, the usual axiomatization of set theory, proves the existence of such objects as stationary and co-stationary subsets of $\omega_1$ (or $\aleph_1$-Aronszajn trees, or well-orders of the reals, etc.), but does not provide any obvious way of finding instances of such objects which are definable.[3] The power of *ZFC* in deriving the existence of those objects comes mainly from its non-constructive set-existence axiom, namely the Axiom of Choice. We may say that the Axiom of Choice is a non-constructive set-existence axiom in that it asserts the existence of certain sets without actually giving a procedure for constructing them. More precisely, the Axiom of Choice guarantees, for a given set $X$, that a set $Y$ satisfying some property $P(X, Y)$ exists, but does not provide an actual definition, from $X$, of any object $Y$ such that $P(X, Y)$ holds. We may say that such a set $Y$ is a non-constructive object relative to *ZFC*. To give an example of a constructive set-existence axiom, note that, on the other hand, the Power Set Axiom asserts the existence, for any given set $X$, of the unique power set of $X$, which of course is definable from $X$. [4]

For example, *ZFC* proves that there is a well-order of the reals and, on the other hand, as the following formulation of a classical result of Cohen shows, one can always build a set-forcing extension $M$ of the universe (so *ZFC* holds in $M$) in which there is no definable well-order of the reals (even allowing real numbers as parameters).

**Theorem 1.1** (*Cohen*). *Let $\kappa$ be an infinite cardinal, let $\mathcal{P}$ be the finite-support product of $\kappa$-many copies of Cohen forcing $2^{<\omega}$ and let $G$ be a $\mathcal{P}$-generic filter over the ground model $V$. Then, in $V[G]$ there is no well-order $\leq$ of the reals such that $\leq$ is definable, in $V[G]$, with parameters from $V(H_\kappa^{V[G]})$, where $V(H_\kappa^{V[G]})$ is the ZF-model $\bigcup_{\alpha\in Ord} L(V_\alpha \cup H_\kappa^{V[G]})$.*

There are certainly models of *ZFC* in which non-constructive objects, relative to *ZFC*, are definable. For example, Gödel proved that in $L$ there is a projective (in fact $\Delta^1_2$) well-order of the reals, and Silver proved that there is a $\Delta^1_3$ well-order of the reals in the minimal model $L[\mu]$ with a measurable cardinal. Of course, both $L$ and $L[\mu]$ are minimal inner cardinals with quite a limited capacity for allocating large cardinals.

[2] In other words, if all its quantifiers occur in a subformula of the form $(\forall x)(x \in y \to \varphi)$ or $(\exists x)(x \in y \wedge \varphi)$.

[3] *Definition* here should obviously be understood as without parameters or at most with *small* parameters, that is with parameters hereditarily of size less than the object being defined.

[4] Of course, the meaning of the term *constructive* as used here is different from the meaning of *constructible* in the sense of belonging to the constructible universe ($L$).