

Computing interpolants in implicational logics

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Abstract

I present a new syntactical method for proving the Interpolation Theorem for the implicational fragment of intuitionistic logic and its substructural subsystems. This method, like Prawitz's, works on natural deductions rather than sequent derivations, and, unlike existing methods, always finds a 'strongest' interpolant under a certain restricted but reasonable notion of what counts as an 'interpolant'.

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1. Introduction

This work is motivated by the following problem in the simply typed λ -calculus:

Problem. Given a normal term $T[\vec{x}, \vec{y}]$, find two normal terms $S[\vec{x}]$ and $P[z, \vec{y}]$ such that $P[S[\vec{x}], \vec{y}] \rightarrow_{\beta} T[\vec{x}, \vec{y}]$.

The question of how to solve this problem occupies a central place in a certain computational model of acquisition of word meanings by children [7,8].¹ Finding one solution to this problem is easy; any instance of the problem has the following solution:

$$\begin{aligned} S[\vec{x}] &= \lambda w. w\vec{x}, \\ P[z, \vec{y}] &= z(\lambda \vec{x}. T[\vec{x}, \vec{y}]). \end{aligned}$$

However, there are many other solutions, and one particularly interesting class of solutions, from the standpoint of the computational model mentioned above, consists of those solutions that assign a 'simplest' type to S .

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¹ We cannot go into any details in this paper, but very briefly, the model assumes that meanings of words and sentences, as well as ways of combining word meanings to build sentence meanings (called "meaning recipes"), are represented by typed λ -terms. Meanings of words and sentences contain constants that represent "semantic primitives", but meaning recipes are pure λ -terms without constants. Suppose that a child encounters a sentence whose meaning $T[\vec{c}, \vec{d}]$ (with constants $\vec{c}, \vec{d}$) is clear to her but which contains one word new to her. If she can tell that constants $\vec{c}$ come from the unknown word and $\vec{d}$ come from the rest of the sentence, then finding out the meaning of the unknown word consists in finding an appropriate pair of terms $S[\vec{x}], P[z, \vec{y}]$ such that $P[S[\vec{x}], \vec{y}] \rightarrow_{\beta} T[\vec{x}, \vec{y}]$.

It turns out that standard syntactical proofs of the *Interpolation Theorem* for intuitionistic logic provide algorithms for finding such solutions. There are two well-known syntactical methods for proving the Interpolation Theorem, one given by Maehara [9] (see [16] for the history and details of the method) and one given by Prawitz [15]. Maehara's method works by induction on cut-free sequent derivations, and Prawitz's method works by induction on normal natural deductions. In these methods, what is to be proved by induction is the following statement:

Interpolation Theorem. *If $\vdash \Gamma, \Delta \Rightarrow C$, then there is a formula E such that*

- $\vdash \Gamma \Rightarrow E$;
- $\vdash E, \Delta \Rightarrow C$;
- *all propositional variables in E appear both in Γ and in Δ, C .*

A formula E satisfying the above conditions is called an *interpolation formula* to the sequent $\Gamma, \Delta \Rightarrow C$ with respect to the *partition* $(\Gamma; \Delta)$ of its antecedent. Implicit in the inductive proof of this statement is an algorithm that, given a cut-free derivation/normal deduction $\mathcal{D}: \Gamma, \Delta \Rightarrow C$, finds two cut-free derivations/normal deductions $\mathcal{D}_1: \Gamma \Rightarrow E$ and $\mathcal{D}_0: E, \Delta \Rightarrow C$. Crucially, the two derivations/deductions $\mathcal{D}_1$ and $\mathcal{D}_0$ found by these methods in fact satisfy much stronger properties. Assuming that $\Gamma, \Delta \Rightarrow C$ consists of implicational formulas, let $T[\vec{x}, \vec{y}]$, $S[\vec{x}]$, $P[z, \vec{y}]$ be the λ -terms corresponding to $\mathcal{D}$, $\mathcal{D}_1$, $\mathcal{D}_0$, respectively, where the types of the variables $\vec{x}$ are the formulas in Γ , the types of the variables $\vec{y}$ are the formulas in Δ , and the type of z is E . Then one has:

- (i) $P[S[\vec{x}], \vec{y}] \rightarrow_{\beta} T[\vec{x}, \vec{y}]$.
- (ii) Both in $\mathcal{D}_1: \Gamma \Rightarrow E$ and in $\mathcal{D}_0: E, \Delta \Rightarrow C$, no occurrence of a propositional variable inside E is *linked* to another such occurrence or originates in an application of Weakening.

Condition (ii) is stated in terms of sequent calculus. In a cut-free sequent derivation, two occurrences of a propositional variable in the endsequent are *linked* to each other if they originate ‘opposite to’ each other in an initial sequent. The condition is invariant across cut-free derivations corresponding to the same normal natural deduction that are *W-normal* in the sense of Mints [10], and it can be stated directly in terms of natural deduction as well. So (ii) is a property of the λ -terms S , P . Condition (i) is emphasized by Čubrić [3] for Prawitz's method, and condition (ii) is a strengthening of one of the conditions stated by Carbone [2] in terms of sequent calculus.

Deviating from standard terminology, we say that a normal term $S[\vec{x}]$ is an *interpolant* to a normal term $T[\vec{x}, \vec{y}]$ (with respect to the partition $(\vec{x}; \vec{y})$ of its free variables) if there exists a normal term $P[z, \vec{y}]$ such that S , P satisfy the conditions (i), (ii). The condition (i) simply says that S , P gives a solution to an instance T of our problem. The condition (ii) gives a sense in which E is ‘simplest’. It implies that in $\mathcal{D}_1$ and $\mathcal{D}_0$, any occurrence of a propositional variable inside E must be linked to an occurrence outside E , from which the third condition on E in the above statement of the Interpolation Theorem follows.

There are two complications, however. One complication is that the Interpolation Theorem in fact fails to hold in the above form for the implicational fragment of intuitionistic logic, which corresponds to the simply typed λ -calculus. Even when $\Gamma, \Delta \Rightarrow C$ is a sequent consisting entirely of implicational formulas, the interpolation formula E sometimes has to contain conjunction. An example of such a sequent is

$$p_1, p_1 \rightarrow p_2, p_1 \rightarrow p_3, p_2 \rightarrow p_3 \rightarrow p_4 \Rightarrow p_4.$$

$p_2 \wedge p_3$ is an interpolation formula to this sequent with respect to the partition $(p_1, p_1 \rightarrow p_2, p_1 \rightarrow p_3; p_2 \rightarrow p_3 \rightarrow p_4)$ of its antecedent, but there is no interpolation formula in the implicational fragment.²

A way of circumventing this problem has been proposed by Wroński [17]. His idea is to use a sequence of formulas $E_1, \dots, E_m$ in place of a single formula E in the statement of the Interpolation Theorem. Although Wroński used this idea to prove an Interpolation Theorem for *BCK*-logic, it can readily be extended to the implicational fragment of intuitionistic logic.³ Thus, we have

² In relation to this, the condition (ii) must be restated in a somewhat weaker form when the interpolation formula is allowed to contain conjunction. In sequent calculus, the present formulation of (ii) can be maintained by adopting a multiplicative version of $(\wedge \Rightarrow)$ in place of Gentzen's [5] rules in *LJ*.

³ Pentus [13] used the same method to prove interpolation for the product-free Lambek calculus.

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