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Computing coproducts of finitely presented Gödel algebras

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Abstract

We obtain an algorithm to compute finite coproducts of finitely generated Gödel algebras, i.e. Heyting algebras satisfying the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$. (Since Gödel algebras are locally finite, 'finitely generated', 'finitely presented', and 'finite' have identical meaning in this paper.) We achieve this result using ordered partitions of finite sets as a key tool to investigate the category opposite to finitely generated Gödel algebras (forests and open order-preserving maps). We give two applications of our main result. We prove that finitely presented Gödel algebras have free products with amalgamation; and we easily obtain a recursive formula for the cardinality of the free Gödel algebra over a finite number of generators first established by A. Horn.

Keywords: Heyting algebras; Gödel algebras; Coproducts; Open maps; Trees; Forests; Ordered partitions

1. Introduction

A *Gödel algebra* (a.k.a. an *L-algebra*, or *Gödel–Dummett algebra*) is a Heyting algebra satisfying the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$. For background and references on Gödel logic and algebras we refer to [4]; for background on Heyting algebras, we refer to [7,3].

Given a variety (i.e., equationally definable class) of algebras V, let V_{fp} denote the full subcategory of finitely presented objects. Informally, a Stone-type duality for V_{fp} amounts to an explicit description of some category C that is dually equivalent to V_{fp}. Evidently, not all such dualities are equally informative: taking $\mathsf{C} = \mathsf{V}_{\mathsf{fp}}^{\mathsf{op}}$, for instance, yields virtually no new insight. It has been authoritatively maintained [8,3] that a reasonable benchmark for the usefulness of a Stone-type duality is the degree to which it affords computation of limits, and thus of colimits in the original category V_{fp}.

The variety of Gödel algebras is locally finite and has finite signature, whence the finitely presentable algebras coincide with the finite and the finitely generated ones. Starting from the standard duality between finite posets and finite distributive lattices, it is a simple matter to develop an effective Stone-type duality for finite Gödel algebras; this we do in Section 2. We marshal enough information on the dual category to eventually establish, in Section 3, an algorithm to compute finite coproducts of finite Gödel algebras in terms of ordered partitions of finite sets. Our results also allow computation of fibred coproducts.

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In Section 4, we give two applications. It is known that Gödel algebras have the amalgamation property [9]. We prove that finitely presented Godel algebras in fact have free products with amalgamation (Corollary 4.1). Horn [5] established a recursive formula for the cardinality of the free Gödel algebra over n generators. We reobtain Horn's formula as an easy application of our main results (Corollary 4.2).

Notation. We let $\mathbb{N} = \{1, 2, \ldots\}$. We write $|A|$ for the cardinality of the set A, and $\setminus$ to denote set-theoretic difference. We write $f \upharpoonright A$ to denote the restriction of the function f to A. We let G denote the category of Gödel algebras, and G_{fp} the full subcategory of finitely presented (equivalently, finite, or finitely generated) algebras. We let $\mathcal{G}_n$ denote the free Gödel algebra over n generators.

2. Preliminaries: Duals of finitely presented Gödel algebras

For general Stone-type dualities see [7]. On the basis of [5, Theorem 2.4], we provide an explicit description of $\mathsf{G}_{\mathsf{fp}}^{\mathsf{op}}$ in terms of forests and open order-preserving maps. As is standard, by a *chain* we throughout mean a totally ordered set.

Definition 2.1. A *forest* is a finite poset F such that for every $x \in F$, the set $\{y \in F \mid y \leq x\}$ is a chain when endowed with the order inherited from F. If $S \subseteq F$, the *down-set* of S is[1] $\downarrow S = \{x \in F \mid x \leq y,\ \text{for some}\ y \in S\}$. A *tree* is a forest with a bottom element, called the *root* of the tree. An order-preserving map $f : F \to G$ between forests is *open* iff it carries down-sets to down-sets — for every $S \subseteq F$, $f(\downarrow S) = \downarrow T$ for some $T \subseteq G$. We let F denote the category of forests and open order-preserving maps, and T the full subcategory of trees.

Remark 1. It is easy to see that the condition $f(\downarrow S) = \downarrow T$ above may be replaced by $f(\downarrow x) = \downarrow f(x)$ for every $x \in F$.

The following is a version of the standard duality between finite posets and finite distributive lattices (see e.g. [1]) for finite Gödel algebras:

Proposition 2.2. *The categories* G_{fp} *and* F *are dually equivalent via the functor* **Spec** *that sends a finite Gödel algebra to the poset of its prime filters* (*ordered by reverse*[2] *set-theoretic inclusion*), *and a morphism* $f : A \to B$ *of algebras to the order-preserving map given by*

$$\mathbf{Spec}(f) : \mathfrak{p} \in \mathbf{Spec}(B) \ \mapsto\ \{a \in A \mid f(a) \in \mathfrak{p}\} \in \mathbf{Spec}(A)\ .$$

Proof. A straightforward verification. To check that duals of finite Gödel algebras are precisely forests, one uses the easily proved fact that the prime filters of a finite (in fact, by [5, Theorem 2.4], of any) Gödel algebra form a forest under reverse set-theoretic inclusion. Moreover, a Heyting algebra whose prime filters form a forest is necessarily a Gödel algebra. It is well-known that order-preserving maps preserve implication iff they are open — for details see for instance [3]. □

The terminal object in F (and T) is a tree consisting of the root only. Finite coproducts in F (and T) are disjoint unions. Finite (fibred) products in F exist because G is a variety; a moment's reflection shows that T has finite products, and a finite product computed in T coincides with the same product computed in F.

Notation. We use $\prod$ and $\coprod$ for products and coproducts, respectively; we also write $X \times Y$ for the product of two objects, and $X \times_Z Y$ for the product of X and Y fibred over Z. We identify without further warning a finite set $\{T_1, \ldots, T_n\}$ of trees with the forest F whose trees are precisely the T_i's — in other words, $F = \coprod_{i=1}^{n} T_i$. We let $\mathcal{C}_n$ denote (the isomorphism type in F of) a chain of cardinality $n + 1$, for $n \geq 0$. We let $\mathcal{S} = \{\mathcal{C}_0, \mathcal{C}_1\}$.

A trivial computation shows that $\mathcal{G}_1$ is the lattice-theoretic product of two chains of respective lengths 2 and 3, whence $\mathbf{Spec}(\mathcal{G}_1) = \mathcal{S}$. Thus, $\mathbf{Spec}(\mathcal{G}_n) = \prod_{i=1}^{n} \mathcal{S} = \mathcal{S}^n$ for any $n \in \mathbb{N}$. Fig. 1(a) shows $\mathcal{S}$. Fig. 1(b), as proved in Section 3, is a picture of $\mathcal{S}^2$.

[1] Following widespread usage, we write $\downarrow x$ for $\downarrow \{x\}$.

[2] A convention herein adopted to make trees grow upwards.