

Non-commutative proof construction: A constraint-based approach

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Abstract

This work presents a computational interpretation of the construction process for cyclic linear logic (CyLL) and non-commutative logic (NL) sequential proofs. We assume a proof construction paradigm, based on a normalisation procedure known as *focussing*, which efficiently manages the non-determinism of the construction.

Similarly to the commutative case, a new formulation of focussing for NL is used to introduce a general constraint-based technique in order to deal with partial information during proof construction. In particular, the procedure develops through construction steps propagating constraints in intermediate objects called *abstract proofs*.

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1. Introduction

1.1. The proof construction paradigm

We are interested here in the computational paradigm of proof construction in logical sequent calculi. The most straightforward (and naive) proof construction algorithm starts with a single open node labelled by a given sequent (or even a single formula), then tries to incrementally construct a proof by repeatedly expanding each open node, selecting and applying an inference from the sequent calculus, thus possibly introducing new open nodes. This provides an interesting computational model, particularly adapted to capture non-deterministic processes, since proof construction itself is intrinsically non-deterministic: in the naive procedure, for example, many choices of different kinds (choice of the principal formula, choice of the inference) have to be made at each expansion step.

1.2. Focussing strategy and bipolar sequent calculus

However, it is well known that, due to intrinsic permutability properties of the inferences in sequent calculi, some strategy is needed in order to avoid making sets of choices that lead to the same object (modulo permutation of

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inferences). Such a strategy, called focussing, has been proposed in [2]. It is based on the generic concept of polarity of formulas, and therefore applies to any logical system where connectives have polarities, such as linear logic or non-commutative logic (see Section 2 for an introduction to non-commutative logic). Focussing deals with two important forms of irrelevant non-determinism in proof construction: on the one hand, the *instant* of the decomposition of a *negative* connective is simply irrelevant; on the other hand, the *interval* between two decompositions of *positive* connectives, if one is an immediate successor of the other in a formula, is irrelevant. The strategy to avoid these forms of irrelevant non-determinism can be expressed in the sequent calculi themselves by modifying the syntax of the sequents, with the introduction of a distinguished formula called the “focus”. Such focussing sequent calculi have thus been proposed in various contexts [17]. There is also an alternative presentation that does not rely on syntactic conventions, while capturing exactly the same content: keep sequents as simple as possible (e.g., they are made of atoms only), but refine the inferences of the calculus themselves. Such a refined, simplified calculus, called the (focussing) *bipolar* calculus, has been presented in [3] for linear logic. It is strictly isomorphic to the focussing sequent calculus of linear logic, so proof construction can be performed equivalently in both systems. Here we extend the focussing bipolar calculus to non-commutative logic without problem, given the genericity of the approach.

1.3. Dealing with partial information

Now, the naive proof construction procedure can be directly applied to the focussing bipolar calculus, but this is still unsatisfactory. Indeed, although focussing eliminates a lot of irrelevant non-determinism, there still remains sources of non-determinism that are intractable in the naive approach. The most well-known one appears in the first-order case with the existential quantification rule [18]: a choice of a term to assign to the quantified variable has to be made, but there are infinitely many possibilities. A random enumeration is clearly unsuitable, essentially because two different random choices may lead to essentially the same object, differing only by a term which, anyway, is irrelevant to the proof. For example, here are two proofs of the formula $\forall x \, p(x) \multimap \exists x \, p(x)$ which differ by a purely irrelevant choice (of the term to assign to x):

$$\frac{\frac{\frac{}{\vdash p(a), p(a)^\perp}}{\vdash \exists x \, p(x), p(a)^\perp} \exists}{\vdash \exists x \, p(x), \exists x \, p(x)^\perp} \exists \quad \frac{\frac{\frac{}{\vdash p(b), p(b)^\perp}}{\vdash \exists x \, p(x), p(b)^\perp} \exists}{\vdash \exists x \, p(x), \exists x \, p(x)^\perp} \exists$$

Again, we need a strategy to avoid such irrelevant non-determinism. In the case of the existential quantification, the solution is quite familiar: it is based on unification, which performs the choice of the term for an existentially quantified variable in a lazy way, making actual choices only when they are needed somewhere in the proof. This leads the construction procedure to manipulate not ground proofs but abstract proofs containing only partial information about the object being constructed (partiality derives from the presence of uninstantiated variables occurring in the terms). This also means that, at each step, some constraints on the different variables have to be propagated across the proof, when variables are shared between multiple nodes: this is the role of the unification process. We thus obtain a slightly less naive construction process which intertwines the usual naive expansion process and the “dual” unification process, which works in the reverse direction.

There is another case of irrelevant non-determinism that the naive procedure cannot avoid, namely caused by the positive multiplicative connectives: to decompose such a connective, a choice of splitting of the context has to be performed. And again, as before, a random enumeration of all the possible splits is unfeasible. A solution based on lazy splitting [9], and capable of dealing with partially defined (or even, possibly, completely undefined) sequents, has been proposed for the focussing bipolar calculus of linear logic in [3]. It couples the naive construction procedure with a non-deterministic but finite constraint propagation procedure, thus leading to a constraint based construction mechanism.

1.4. Non-commutative logic

Here we study the extension of this mechanism to non-commutative logic, NL. It is shown that it extends straightforwardly to cyclic logic, but not directly to non-commutative logic, although NL is an extension of both linear and cyclic logic. We then propose a new solution, which works on an affine version of NL where *weakening*

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