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Non-primitive recursive decidability of products of modal logics with expanding domains

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Abstract

We show that—unlike products of 'transitive' modal logics which are usually undecidable—their 'expanding domain' relativisations can be decidable, though not in primitive recursive time. In particular, we prove the decidability and the finite expanding product model property of bimodal logics interpreted in two-dimensional structures where one component—call it the 'flow of time'—is

• a finite linear order or a finite transitive tree

and the other is composed of structures like

• transitive trees/partial orders/quasi-orders/linear orders or only finite such structures

expanding over time. (It is known that none of these logics is decidable when interpreted in structures where the second component does not change over time.) The decidability proof is based on Kruskal's tree theorem, and the proof of non-primitive recursiveness is by reduction of the reachability problem for lossy channel systems. The result is used to show that the dynamic topological logic interpreted in topological spaces with continuous functions is decidable (in non-primitive recursive time) if the number of function iterations is assumed to be finite.

Keywords: Modal logic; Temporal logic; Decidability; Dynamic topological logic

1. Introduction

Started in the 1970s [40,41], the research programme of investigating and using *products of modal logics*[1] as a multi-dimensional formalism for a variety of promising applications in mathematical logic, computer science and

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[1] For the definition of *products* of modal logics see Section 5 below.

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artificial intelligence (see, e.g., [2,36,9,4,37,13,7,45]) has recently culminated in a series of interesting decidability and complexity results.

Decidability: Roughly, a two-dimensional product of modal logics can be decidable only if, in order to check satisfiability of a formula φ in product frames for the logic, it suffices to consider those of them where the depth of one of the component frames is bounded by some finite number depending on φ. In other words, only products of standard modal logics with **K**-like or **S5**-like[2] logics are decidable [13,44,11]. Three-dimensional products and products of transitive logics with arbitrary finite or infinite frames are *not* decidable [31,17,38,14].

Complexity: The computational complexity of decidable product logics turns out to be much higher than the complexity of their components. For example, it is shown in [32] that *all* product logics between $\mathbf{K} \times \mathbf{K}$ and $\mathbf{S5} \times \mathbf{S5}$ are coNExpTime-hard (while **K** is known to be PSpace-complete and **S5** coNP-complete). According to [33], even the satisfiability problem for formulas of modal depth 2 in $\mathbf{K} \times \mathbf{K}$-frames is NExpTime-hard. $\mathsf{Log}(\mathbb{N}, <) \times \mathbf{S5}$ is ExpSpace-hard, while $\mathbf{PTL} \times \mathbf{K}$ is not elementary [16,18,11].

Such is the price we have to pay for the strong interaction between the modal operators of the component logics of a product, which is syntactically reflected by the (seemingly harmless) commutativity and Church–Rosser axioms

$$\Diamond_1 \Diamond_2 p \leftrightarrow \Diamond_2 \Diamond_1 p \qquad \text{and} \qquad \Diamond_2 \Box_1 p \to \Box_1 \Diamond_2 p.$$

The general research problem we are facing now can be formulated as follows: *is it possible to reduce the computational complexity of product logics by relaxing the interaction between their components and yet keeping some of the useful and attractive features of the product construction*?

One approach to this problem is motivated by structures often used in such areas as temporal and modal first-order logics, temporal data or knowledge bases (say, temporal description logics) or logical modelling of dynamical systems. What we mean is models/structures with *expanding domains*: if at a certain time point (or in a world) w we have a 'population' Δ_w of elements (objects), then at every later point (in every accessible world) u the population Δ_u cannot be smaller but can grow—i.e., $\Delta_w \subseteq \Delta_u$. Standard product logics respect the stronger *constant domain assumption* according to which $\Delta_w = \Delta_u$ for all u and w.

In the case of *dynamic topological logics* [27,21], expanding domains correspond to the condition that the function describing movements of points in topological spaces is *continuous* (while constant domains correspond to *homeomorphisms*).

Models with expanding domains naturally arise also in the context of tableau- and resolution-based decision procedures that have been developed and implemented for certain monodic fragments of first-order temporal logic and some modal description logics [15,24,20] which include, in particular, the (expanding) products of the corresponding temporal and modal logics with **S5**. One of the most difficult problems in the development and implementation was the conflict between modularity and the necessity to backtrack after introducing every new element; in fact, the systems developed so far are considerably more efficient for expanding domain than for constant domain interpretations.

Products of modal logics with expanding domains were introduced in [30], where it was shown that they cannot be more complex than (in fact, are reducible to) products. But can they be simpler? For example, is it possible that a product logic is undecidable while its expanding relativisation is decidable? A similar question was asked in [12] where it was shown that the two-variable fragment of most first-order modal logics with constant domains is undecidable.

The main achievement of this paper is the discovery of the first pairs of 'standard' modal logics whose product with expanding domains is indeed simpler than their usual product. For example, we show that the expanding product of **GL.3** and **GL** is decidable and has the expanding product finite model property—in contrast to the product $\mathbf{GL.3} \times \mathbf{GL}$ which is undecidable and does not even have the (abstract) finite model property [14]. As a consequence of our results on expanding products, we also prove that the dynamic topological logic with continuous functions and finitely many iterations is decidable—again in contrast to the undecidability in the case of dynamic topological structures with homeomorphisms [21].

Our main results can be summarised as follows. Bimodal logics interpreted in expanding product frames where the first component consists of

[2] The definitions of some standard modal logics like **K**, **S5**, etc., can be found in Section 2.