

Models of real-valued measurability

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Abstract

Solovay's random-real forcing [R.M. Solovay, Real-valued measurable cardinals, in: Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967), Amer. Math. Soc., Providence, R.I., 1971, pp. 397–428] is the standard way of producing real-valued measurable cardinals. Following questions of Fremlin, by giving a new construction, we show that there are combinatorial, measure-theoretic properties of Solovay's model that do not follow from the existence of real-valued measurability.

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1. Introduction

Solovay [8] showed how to produce a real-valued measurable cardinal by adding random reals to a ground model which contains a measurable cardinal. (Recall that a cardinal κ is *real-valued measurable* if there is an atomless, κ -additive measure on κ that measures all subsets of κ . For a survey of real-valued measurable cardinals see Fremlin [4].)

The existence of real-valued measurable cardinals is equivalent to the existence of a countably additive measure on the reals which measures all sets of reals and extends Lebesgue measure (Ulam [9]). However, the existence of real-valued measurable cardinals, and particularly if the continuum is real-valued measurable, has an array of Set Theoretic consequences reaching beyond measure theory. For example: a real-valued measurable cardinal has the tree property (Silver [7]); if there is a real-valued measurable cardinal, then there is no rapid p -point ultrafilter on $\mathbb{N}$ (Kunen); the dominating invariant $\mathfrak{d}$ cannot equal a real-valued measurable cardinal (Fremlin). And further, if the continuum is real-valued measurable then $\diamond_{2^{\aleph_0}}$ holds (Kunen); and for all cardinals λ between $\aleph_0$ and the continuum we have $2^\lambda = 2^{\aleph_0}$ (Prikry [6]); see [4].

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On the other hand, there are other properties of Solovay's model that have not been shown to follow from the mere existence of real-valued measurable cardinals: for example, the covering invariant for the null ideal $\text{cov}(\mathcal{N})$ has to equal the continuum.

Thus, Fremlin asked [4, P1] whether every real-valued measurable cardinal can be obtained by Solovay's method (the precise wording is: suppose that κ is real-valued measurable; must there be an inner model $M \subset V$ such that κ is measurable in M and a random extension $M[G] \subset V$ of M which contains $\mathcal{P}\kappa$?). The question was answered in the negative by Gitik and Shelah [5]. The broader question remains: what properties of Solovay's model follow from the particular construction, and which properties are inherent in real-valued measurability?

In this paper we present a new construction of a real-valued measurable cardinal and identify a combinatorial, measure-theoretic property that differentiates between Solovay's model and the new one.

The property is the existence of what we call *general sequences* — Definition 4.5. A general sequence is a sequence which is sufficiently random as to escape all sets of measure zero. Standard definitions of randomness are always restricted, in the sense that the randomness has to be measured with respect to a specified collection of null sets (from effective Martin-Löf tests to all sets of measure zero in some ground model). Of course, we cannot simply remove all restrictions, as no real escapes all null sets. However, we are interested in a notion that does not restrict to a special collection of null sets but considers them all. One way to do this is to change the nature of the random object — here, from a real to a long sequence of reals, and to change the nature of escaping. We remark here that the definition echoes (in spirit) the characterization of (effective) Martin-Löf randomness as a string, each of whose initial segments have high Kolmogorov complexity.

We thus introduce a notion of forcing $\mathbb{Q}_\kappa$. We show that if κ is measurable (and $2^\kappa = \kappa^+$), then in $V^{\mathbb{Q}_\kappa}$, κ (which is the continuum) is real-valued measurable (Theorem 3.18). We then show that in Solovay's model, the generic (random) sequence is general (Theorem 4.6); and that in the new model, no sequence is general (Theorem 4.14).

1.1. Notation

$\mathcal{P}X$ is the power set of X . $A - B$ is set difference. $\subset$ denotes inclusion, not necessarily proper; $\subsetneq$ denotes proper inclusion.

The reals $\mathbb{R}$ are identified with Cantor space 2^ω . If $\sigma \in 2^{<\omega}$ then $[\sigma] = \{x \in \mathbb{R} : \sigma \subset x\}$ denotes the basic open set determined by σ . If $\lambda \in \text{On}$ then $\mathbb{R}^\lambda$ is the λ -fold product of $\mathbb{R}$. If $\alpha < \lambda$ and B is a Borel subset of $\mathbb{R}$ then B^α denotes $\{\bar{x} \in \mathbb{R}^\lambda : x_\alpha \in B\}$.

If A is a Borel set (on some copy of Cantor space) and W is an extension of the universe V then we let A^W denote the interpretation in W of any code of A .

If $\mathbb{P} = (\mathbb{P}, \leq)$ is a partial ordering then we sometimes write $\leq_{\mathbb{P}}$ for $\leq$.

If $\alpha < \beta$ are ordinals then $[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}$.

If X and Y are sets and $B \subset X \times Y$, then for $x \in X$, $B_x = \{y \in Y : (x, y) \in B\}$ and $B^y = \{x : (x, y) \in B\}$ are the sections.

Suppose that $\langle X_\alpha \rangle_{\alpha < \delta}$ is an increasing sequence of things (ordinals, sets (under inclusion), etc.); for limit $\beta \leq \delta$ we let $X_{<\beta}$ be the natural limit of $\langle X_\alpha \rangle_{\alpha < \beta}$ (the supremum, the union, etc.), and for successor $\beta = \alpha + 1$ we let $X_{<\beta} = X_\alpha$.

1.1.1. Forcing

For notions of forcing, we use the notation common in the World — {Jerusalem}. Thus, $q \leq p$ means that q extends p . As far as $\mathbb{P}$ -names are concerned, we often confuse between canonical objects and their names. Thus, G is both a generic filter but also the name of such a filter.

If $\mathbb{B}$ is a complete Boolean algebra and φ is a formula in the forcing language for $\mathbb{B}$, then we let $\llbracket \varphi \rrbracket_{\mathbb{B}}$ be the Boolean value of φ according to $\mathbb{B}$; this is the greatest element of $\mathbb{B}$ forcing φ . For a complete Boolean algebra the partial ordering corresponding to $\mathbb{B}$ is not $\mathbb{B}$ itself but $\mathbb{B} - \{0_{\mathbb{B}}\}$. Nevertheless we often think as if the partial ordering in the forcing were $\mathbb{B}$ and let $0 \Vdash_{\mathbb{B}} \varphi$ for all formulas φ in the forcing language.

If $\mathbb{P}$ is a partial ordering and $p \in \mathbb{P}$ then $\mathbb{P}(\leq p)$ is the partial ordering inherited from $\mathbb{P}$ on $\{q \in \mathbb{P} : q \leq p\}$.

$\mathbb{P} \triangleleft \mathbb{Q}$ denotes the fact that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$. If $\mathbb{P} \triangleleft \mathbb{Q}$ and G is the (name for the) $\mathbb{P}$ -generic filter, then $\mathbb{Q}/G$ is the (name for the) quotient of $\mathbb{Q}$ by G : the collection of all $q \in \mathbb{Q}$ which are compatible with all $p \in G$.

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