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The saturation of club guessing ideals

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Abstract

We prove that it is consistent that there exists a saturated tail club guessing ideal on ω_1 which is not a restriction of the non-stationary ideal. Two proofs are presented. The first one uses a new forcing axiom whose consistency can be proved from a supercompact cardinal. The resulting model can satisfy either CH or $2^{\aleph_0} = \aleph_2$. The second one is a direct proof from a Woodin cardinal, which gives a witnessing model with CH.

Keywords: Club guessing; Saturated ideal; Forcing axiom; Supercompact; Woodin; Revised countable support iteration

0. Introduction

The notion of club guessing sequences was introduced by Shelah. Some of its earliest appearances are found in [12] and [15, III]. The same notion on ω_1 was also considered by Nyikos in [14]. It has been used fruitfully in set theory. For example, PCF theory relies on the notion and Gitik and Shelah proved that the non-stationary ideal on a regular cardinal above $\aleph_1$ cannot be saturated by investigating certain club guessing sequences. More applications have been found recently.

Associated with a club guessing sequence are the tail club guessing filter and the fully club guessing filter. Their dual ideals are called the tail club guessing ideal and the fully club guessing ideal respectively. They, too, were introduced and studied by Shelah in [15]. Club guessing ideals are not only good indicators of the behavior of club guessing sequences, but also intrinsically interesting objects. A club guessing ideal is closely related to the non-stationary ideal and thus shares many properties. However, by taking different club guessing sequences, we can obtain various club guessing ideals. Hence, we may construct a club guessing ideal which is significantly different from the non-stationary ideal. Such results will increase our understanding on the non-stationary ideal.

Large cardinal properties of ideals, such as precipitousness and saturation, interested many researchers in set theory. This notion was first applied by Solovay in [17]. Assuming the existence of a real-valued measurable cardinal, he constructed an elementary embedding via forcing. An elementary embedding constructed by the same technique is called a generic embedding. In [9], Jech and Prikry studied the ideals which can be used to build a generic embedding. Such ideals were called precipitous ideals. The existence of precipitous ideals together with other properties of the universe gives many consequences, which were exemplified in [10].

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When an ideal is saturated as in Solovay's proof, the generic embedding behaves well. Let I be an ideal on an uncountable regular cardinal κ. If $\mathcal{P}(\kappa)/I$ is λ-cc, we say that I is λ-saturated. When I is κ^+-saturated, we say that I is saturated. It is essentially proved by Solovay in [17] that every saturated ideal is precipitous. Thus saturatedness is also considered as a large cardinal property.

It is particularly interesting when a naturally defined ideal has a large cardinal property because its natural definition gives more information about the generic embedding. For example, if a tail club guessing ideal is precipitous, then the generic embedding produces a non-trivial instance of the outside club guessing phenomena, studied by Džamonja and Shelah in [4]. Thus, we are interested in large cardinal properties of club guessing ideals.

Several theorems about large cardinal properties of club guessing ideals were already proved. Woodin showed in [18, Chapter 8] that it is consistent relative to the existence of a Woodin cardinal that there exists a club guessing sequence on ω_1 such that the tail club guessing ideal associated with it is saturated. Foreman and Komjáth proved in [5] that it is consistent relative to the existence of an almost huge cardinal above an uncountable regular cardinal κ that there exists a club guessing sequence on κ such that the tail club guessing ideal associated with it is saturated. However, in these two results, the tail club guessing ideal is a restriction of the non-stationary ideal.

The author proved in [8] that it is consistent relative to the existence of a Woodin cardinal above an uncountable regular cardinal κ that all tail club guessing ideals on κ are precipitous. This result gives a model with many essentially different precipitous ideals.

Nonetheless, it remained open whether a tail club guessing ideal can be saturated without being a restriction of the non-stationary ideal. In this paper, we resolve this question positively, i.e. we shall prove the following theorem.

Theorem. *Suppose that there exists a Woodin cardinal. Then it is consistent that there exists a tail club guessing sequence on ω_1 such that the tail club guessing ideal associated with it is saturated but not a restriction of the non-stationary ideal.*

We shall also provide a forcing axiom which implies the consequence of the previous theorem and prove the consistency of the axiom from a supercompact cardinal.

The proof with a forcing axiom is similar to the argument given by Foreman, Magidor, and Shelah in [6] to show that Martin's Maximum implies that the non-stationary ideal on ω_1 is saturated. However, the tools used in their proofs do not work for tail club guessing ideals. Thus, we need to define several corresponding objects appropriate in this context and prove lemmas regarding them. The following table shows the relationship between the objects in Foreman, Magidor, and Shelah's argument and ours.

Foreman, Magidor, and Shelah	This paper
Stationary preserving forcing	totally $\vec{C}$-preserving forcing
Semiproper forcing	$\vec{C}$-semiproper forcing
The set of all countable elementary substructure of $\mathfrak{A}$	$\tilde{S}(\vec{C}, \mathfrak{A})$
The club filter on $[H(\lambda)]^{\aleph_0}$	$\mathcal{F}(\vec{C}, H(\lambda))$

Shelah showed in [16] that semiproperness is preserved by the revised countable support (RCS) iteration. By a similar argument, we shall show that $\vec{C}$-semiproperness is preserved by the RCS-iteration. Instead of Shelah's original one, we use the exposition by Donder and Fuchs in [2].

We also investigate the κ-saturatedness of club guessing ideals. Solovay showed in [17] that the non-stationary ideal on an uncountable regular cardinal κ cannot be κ-saturated. Similarly, we can show the following theorem.

Theorem. *A tail club guessing ideal on an uncountable regular cardinal κ cannot be κ-saturated.*

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