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Empiricism, probability, and knowledge of arithmetic: A preliminary defense

Sean Walsh

Department of Logic and Philosophy of Science, University of California, 5100 Social Science Plaza, Irvine, CA 92697-5100, USA

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ABSTRACT

The topic of this paper is our knowledge of the natural numbers, and in particular, our knowledge of the basic axioms for the natural numbers, namely the Peano axioms. The thesis defended in this paper is that knowledge of these axioms may be gained by recourse to judgments of probability. While considerations of probability have come to the forefront in recent epistemology, it seems safe to say that the thesis defended here is heterodox from the vantage point of traditional philosophy of mathematics. So this paper focuses on providing a preliminary defense of this thesis, in that it focuses on responding to several objections. Some of these objections are from the classical literature, such as Frege's concern about indiscernibility and circularity (Section 2.1), while other are more recent, such as Baker's concern about the unreliability of small samplings in the setting of arithmetic (Section 2.2). Another family of objections suggests that we simply do not have access to probability assignments in the setting of arithmetic, either due to issues related to the ω-rule (Section 3.1) or to the non-computability and non-continuity of probability assignments (Section 3.2). Articulating these objections and the responses to them involves developing some non-trivial results on probability assignments (Appendix A–Appendix C), such as a forcing argument to establish the existence of continuous probability assignments that may be computably approximated (Theorem 4 Appendix B). In the concluding section, two problems for future work are discussed: developing the source of arithmetical confirmation and responding to the probabilistic liar.

1. Introduction

The topic of this paper is the tenability of a certain type of empiricism about our knowledge of the Peano axioms. The Peano axioms constitute the standard contemporary axiomatization of arithmetic, and they consist of two parts, a set of eight axioms called *Robinson's Q*, which ensure the correctness of the addition and multiplication tables, and the principle of *mathematical induction*, which says that if zero has a given

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property and $n + 1$ has it whenever n has it, then all natural numbers have this property.[1] The type of empiricism about the Peano axioms which I want to consider holds that arithmetical knowledge is akin to the knowledge by which we infer from the past to the future, or from the observed to the unobserved. It is not uncommon today to hold that such inductive inferences can be rationally sustained by appeal to informed judgments of probability. The goal of this paper is to defend an empiricism which contends that judgments of probability can help us to secure knowledge of the Peano axioms.

This empiricism merits our attention primarily because standard accounts of our knowledge of the Peano axioms face difficult problems, problems going above and beyond skepticism about knowledge of abstract objects. For instance, logicism suggests that knowledge of the Peano axioms may be based on knowledge of ostensibly logical principles – such as Hume's Principle – and the knowledge that the Peano axioms are representable within these logical principles (cf. [48] p. xiv, p. 131). The success of logicism thus hinges upon identifying a concept of representation that can sustain this inference, and as I have argued elsewhere, it seems that we presently possess no such concept [45]. Alternatively, some structuralists have suggested that knowledge of the Peano axioms may be based on our knowledge of the class of finite structures. However, this account then owes us an explanation of why the analogues of the Peano axioms hold on the class of finite structures. For example, this account must tell us something about how we know that there's no finite structure that is larger than all the other finite structures (cf. [40] p. 112, [30] p. 159).

The second reason that this kind of empiricism about the Peano axioms merits our attention is that it has been suggested in different ways by both historical and contemporary sources. For instance, prior to Frege, a not uncommon view seems to have been that mathematical induction was an empirical truth akin to enumerative induction. This is why Kästner thought that mathematical induction was not fit to be an axiom ([23] pp. 426–428), and this is part of the background to Reid's begrudging concession that "necessary truths may sometimes have probable evidence" ([38] VII.ii.1).[2] However, some contemporary authors writing on the epistemology of arithmetic and arithmetical cognition have also suggested views related to this. For instance, Rips and Asmuth – two cognitive scientists who work on mathematical cognition – have recently considered the suggestion that "the theoretical distinction between math[ematical] induction and empirical induction" is not as clear as has been claimed, and that "even if the theoretical difference were secure, it

[1] More formally, the axioms of Robinson's Q are the following:

(Q1) $Sx \neq 0$
(Q2) $Sx = Sy \rightarrow x = y$
(Q3) $x \neq 0 \rightarrow \exists\, w\; x = Sw$
(Q4) $x + 0 = x$
(Q5) $x + Sy = S(x + y)$
(Q6) $x \cdot 0 = 0$
(Q7) $x \cdot Sy = x \cdot y + x$
(Q8) $x \leqslant y \leftrightarrow \exists\, z\; x + z = y$

For the formal result that indicates that Robinson's Q ensures the correctness of the addition and multiplication tables (among other things), see Proposition 1 in Section 2.2. The Peano axioms then consist of Robinson's Q along with each instance of the mathematical induction schema, wherein $\varphi(x)$ ranges over first-order formulas with one free variable x:

$$[\varphi(0)\ \&\ \forall\, y\ \varphi(y) \rightarrow \varphi(S(y))] \rightarrow [\forall\, x\ \varphi(x)] \quad (1)$$

Hence, what I am describing in this paper as "the Peano axioms" is first-order Peano arithmetic, as described and studied in e.g. [15]. This is to be distinguished from second-order Peano arithmetic as studied in e.g. [41], wherein the mathematical induction schema is replaced by single induction axiom and in which one additionally adds the comprehension schema, which says that every formula with a free first-order variable determines a second-order entity. Against the background of second-order logic with the comprehension schema, the mathematical induction axiom is equivalent to the version of the mathematical induction schema (1) wherein $\varphi(x)$ is permitted to range over second-order formulas with one free object variable x. Hence it makes no difference to the arguments presented here whether one works in a first- or second-order setting, and so for the sake of simplicity I keep here to the first-order setting.

[2] Mill, by contrast, thought that proofs related to mathematical induction ought not be conceived of as instances of enumerative induction (cf. [34] vol. 7 p. 288 ff, Book III, Chapter 2). The history of this topic obviously deserves more discussion than I am able to give here.