Journal of Applied Logic 12 (2014) 349–368

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

Towards classifying propositional probabilistic logics

Glauber De Bona [a,*], Fabio Gagliardi Cozman [b], Marcelo Finger [a]

[a] *Institute of Mathematics and Statistics, University of Sao Paulo, Brazil*
[b] *Polytechnic School, University of Sao Paulo, Brazil*

ARTICLE INFO

Article history:
Available online 6 February 2014

Keywords:
Probabilistic logic
Expressivity
Computational complexity

ABSTRACT

This paper examines two aspects of propositional probabilistic logics: the nesting of probabilistic operators, and the expressivity of probabilistic assessments. We show that nesting can be eliminated when the semantics is based on a single probability measure over valuations; we then introduce a classification for probabilistic assessments, and present novel results on their expressivity. Logics in the literature are categorized using our results on nesting and on probabilistic expressivity.

1. Introduction

As even a cursory look at the literature reveals, there are many ways to mix probabilities and propositional logic. Indeed, propositional probabilistic logics appear in several fields, from philosophy to economics and artificial intelligence, with real applications from material discovery [14] to model checking [2]. However, there are relatively few proposals for the classification of propositional probabilistic logics based on their expressivity [10,18,36,39].

In this paper we examine two aspects of propositional probabilistic logics that can be used to classify them. First, we examine the nesting of probabilistic operators, which we identify not only as a major syntactic decision, but also as a decision that drives the semantics. Second, we investigate the expressivity of probabilistic assessments, a topic that has received scant attention in the literature but that has interesting consequences.

We restrict our study to propositional probabilistic logics that adopt classical propositional logic as a starting point; we also assume that all models have a finite number of possible worlds, so that issues of countable additivity are irrelevant. Finally, we only deal with "unconditional" probabilities, leaving conditional assessments to future work.[1]

* Corresponding author.
E-mail addresses: debona@ime.usp.br (G. De Bona), fgcozman@usp.br (F.G. Cozman), mfinger@ime.usp.br (M. Finger).

[1] We consider some logics that can make statements constraining the value of $P(\phi \wedge \theta)/P(\theta)$, given $P(\theta) > 0$, in Section 4.3; but this is not the same as constraining the value of $P(\phi|\theta)$ in the general case (in which $P(\theta)$ may be 0).

http://dx.doi.org/10.1016/j.jal.2014.01.005
1570-8683/

After a brief review of relevant concepts in Section 2, we focus on the nesting of probabilistic operators and its semantic consequences in Section 3. That is, we examine the meaning of expressions such as $P(P(\phi) = 0.5) = 0.7$, understood as the "probability of the probability of a propositional formula ϕ being 0.5 is 0.7". We contribute with a novel analysis of the relationship between nesting and semantics. In Section 4 we discuss ways to specify probability assessments; we call it the *probabilistic expressivity* of a logic. For instance, a particular logic may allow one to state $P(\phi) = 1/2$, while another logic may allow $P(\phi_1)^2 + 2P(\phi_2) \leqslant q$ where ϕ_1 and ϕ_2 are propositional formulas and q is restricted to be a rational number. The extent to which one can express probabilistic appraisals strongly affects expressivity and complexity of the whole logic in question.

Some decisions one can take when mixing probabilities and propositions may not have much effect on expressivity and complexity; other decisions can have dramatic effect. Based on our results, we sketch a scheme for classification of propositional probabilistic logics in Section 5.

2. Preliminaries

In this section we offer a brief review of propositional logic and probabilistic satisfiability, so as to fix notation and terminology.

First, we define a logic as a *satisfaction system*, as it is done in [3]:

Definition 1. Let L be a non-empty set (a language) and M be a non-empty class (of models). A *logic* is a tuple $(L, M, \models)$ in which $\models \subseteq M \times L$ is a relation.

2.1. The propositional language and its semantics

The language of propositional logic consists of a set of formulas formed by propositions combined with logical connectives, possibly with punctuation elements. We assume a countably infinite set of symbols $X = \{x_1, x_2, x_3, \dots\}$ corresponding to atomic propositions. We have the unary connective $\neg$ (negation) and the binary connectives $\vee$ (disjunction), $\wedge$ (conjunction) and $\rightarrow$ (implication), plus parentheses (dropped whenever possible). Every proposition is a formula; moreover, if ϕ is a formula, then $\neg\phi$ is a formula; if ϕ_1 and ϕ_2 are formulas, then $(\phi_1 \vee \phi_2)$, $(\phi_1 \wedge \phi_2)$ and $(\phi_1 \rightarrow \phi_2)$ are formulas. The set of all formulas built using only these guidelines is the language of propositional logic, denoted by L_{PL}. Additionally, $\phi_1 \leftrightarrow \phi_2$ (bi-implication) denotes $(\phi_1 \rightarrow \phi_2) \wedge (\phi_2 \rightarrow \phi_1)$, and $\top$ denotes $x_i \vee \neg x_i$ for some x_i.

Each atomic proposition can assume a truth value, either true or false, represented respectively by 1 and 0. A *truth assignment*, or *valuation*, is a function $v : X \rightarrow \{0, 1\}$ that takes atomic propositions to truth values. Valuations may have their domain extended to all of L_{PL}, as follows. Let ϕ_1 and ϕ_2 be formulas from the propositional language; then: $v(\phi_1 \wedge \phi_2) = 1$ if and only if $v(\phi_1) = 1$ and $v(\phi_2) = 1$; $v(\phi_1 \vee \phi_2) = 1$ if and only if $v(\phi_1) = 1$ or $v(\phi_2) = 1$; $v(\neg\phi_1) = 1$ if and only if $v(\phi) = 0$; $v(\phi_1 \rightarrow \phi_2) = 1$ if and only if $v(\phi_1) = 0$ or $v(\phi_2) = 1$. Let V be the set of all valuations over X. The classical propositional logic L_{PL} is the tuple $(L_{PL}, V, \models)$ where $\models \subseteq V \times L_{PL}$ is a relation such that $v \models \phi$ iff $v(\phi) = 1$ for every $(v, \phi) \in V \times L_{PL}$. A propositional formula ϕ is *satisfiable* when it is possible to find a valuation v such that $v \models \phi$.

We will identify a logical formula ϕ with a set of *possible worlds*, to be suitably defined later. Such a set is denoted by $[\![\phi]\!]$. In this paper we always have, for the logics and semantics we define, that $[\![\neg\phi]\!] = \overline{[\![\phi]\!]}$, $[\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \cup [\![\phi_2]\!]$, and $[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!]$.

2.2. Probability theory and probabilistic satisfiability

A probability measure attaches real numbers to events that are subsets of a set Ω, the possibility space. All probability spaces in this paper are finite, hence we can take that any subset of Ω is an event. A probability measure P is such that $P(\Omega) = 1$, $P(A) \geqslant 0$ for any event A, and $P(A \cup B) = P(A) + P(B)$ for any disjoint