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Stochastic λ-calculi: An extended abstract

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A R T I C L E   I N F O

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It is shown how the operators in the "graph model" for λ-calculus (which can function as a programming language for Recursive Function Theory) can be expanded to allow for "random combinators". The result then is a model for a new language for random algorithms.

1. Background

In the mid-1970s the author in [10] defined a "graph model" for the untyped λ-calculus making use of *enumeration operators* from Recursive Function Theory. It turned out that Gordon Plotkin had earlier defined in set-theoretical terms a closely related construction. In 1993 Plotkin published his previously unpublished technical report along with extensive commentary in [8]. A full history of λ-calculus – along with expository material and many, many references – can be found in [2]. *Enumeration operators* had been defined in the mid-1950s independently by Myhill and Shepherdson in [7] and by Friedberg and Rogers in [4]. The book [9] by Rogers popularized the idea of *enumeration degrees*, a subject on which there is now a very large literature. Neither team in [7] or [4] seemed to realize they had sufficient mechanics for defining a model for λ-calculus, however.

Recently several authors have made proposals for defining non-deterministic extensions of the untyped λ-calculus – generally explained using *operational semantics* rather than models for a *denotational semantics*. See [3] and [6] for pro posals and many references. (There are unfortunately too many to mention here in this brief article.) More discussion and references can be found in the very recent papers of Michael Mislove referenced on his web page [1] which also reference other approaches.

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2. Basic definitions

Let $\mathbb{N}$ be the set of *natural numbers*, on which we can use a very easy to define *pairing function*: $(n, m) = 2^n(2m+1)$. This puts the pairs of natural numbers into a one–one correspondence with the *positive integers*. *Sequence numbers* can now be defined as:

$$\langle\,\rangle = 0 \quad \text{and} \quad \langle n_0, n_1, \ldots, n_{k-1}, n_k\rangle = \bigl(\langle n_0, n_1, \ldots, n_{k-1}\rangle, n_k\bigr).$$

This notation puts *finite sequences* of elements of $\mathbb{N}$ into a one–one correspondence with the *whole* of the set $\mathbb{N}$. *Finite sets* of natural numbers can be numbered as follows:

$$\mathbf{set}(0) = \emptyset \quad \text{and} \quad \mathbf{set}\bigl((n, m)\bigr) = \mathbf{set}(n) \cup \{m\}.$$

It then follows that:

$$\mathbf{set}\bigl(\langle n_0, n_1, \ldots, n_{k-1}, n_k\rangle\bigr) = \{n_0, n_1, \ldots, n_{k-1}, n_k\}.$$

There are many other popular ways of numbering (Gödel-numbering) pairs, sequences and sets, but the choice of a numbering system is not so very important as long as the functions are *computable*. And to these notations we add the *Kleene star*: $X^* = \{n \mid \mathbf{set}(n) \subseteq X\}$, which gives the set X^* of all (numbers of) finite sequences of elements of a set $X \subseteq \mathbb{N}$. Note that $\mathbb{N}^* = \mathbb{N}$ and $\emptyset^* = \{0\}$.

Definition 2.1. The *enumeration operators* are identified with the sets F of natural numbers which operate on sets of integers through the binary operation of *application*:

$$F(X) = \bigl\{m \bigm| \exists n \in X^*.\, (n, m) \in F\bigr\}.$$

The idea here is that, as you start enumerating the elements of the set X, you can also then enumerate the elements of the set X^*. Along with these enumerations, you can also enumerate the pairs in F. Every time you see a match between a sequence number $n \in X^*$ and the first term of a pair $(n, m) \in F$, you then *output* m as an element of $F(X)$. The enumerations of the sets F and X thus generate the enumeration of the set $F(X)$.

Except for a small detail in the use of Gödel numbers, this is the same as the definition in [9]. And, following Friedberg and Rogers, we say that a set B is *enumeration reducible* to a set A just in case there is a *recursively enumerable* set F such that $B = F(A)$. The emphasis here is that the *computable* enumeration operators are those given by recursively enumerable sets F.

Next, note that the general enumeration operators have an important connection to *topology*. The powerset

$$\mathcal{P}(\mathbb{N}) = \{X \mid X \subseteq \mathbb{N}\}$$

can be considered as being a topological space with the sets

$$\mathcal{Q}_n = \bigl\{X \mid n \in X^*\bigr\}$$

as a *basis* for the topology. This could be called the *positive topology*, because it works with only the *positive facts* as to which finite sets are in fact contained in a “point” $X \in \mathcal{P}(\mathbb{N})$.

Definition 2.2. We say that a function $\Phi : \mathcal{P}(\mathbb{N})^n \to \mathcal{P}(\mathbb{N})$ of n-arguments is *continuous* in this topology iff (for all integers m) we have: