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Lexicographic pseudo MV-algebras ☆

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A B S T R A C T

A lexicographic pseudo MV-algebra is an algebra that is isomorphic to an interval in the lexicographic product of a unital linearly ordered group with an arbitrary ℓ-group. We present conditions when a pseudo MV-algebra is lexicographic. We show that a key condition is the existence of a lexicographic ideal, or equivalently, a case when the algebra can be split into comparable slices indexed by elements of the interval $[0, u]$ of some unital linearly ordered group (H, u). Finally, we show that fixing (H, u), the category of (H, u)-lexicographic pseudo MV-algebras is categorically equivalent to the category of ℓ-groups.

1. Introduction

MV-algebras are the algebraic counterpart of the infinite-valued Łukasiewicz sentential calculus introduced by Chang in [3]. Perfect MV-algebras were characterized as MV-algebras where each element is either infinitesimal or co-infinitesimal. Therefore, they have no parallels in the realm of Boolean algebras because perfect MV-algebras are not semisimple. The logic of perfect MV-algebras has a counterpart in the Lindenbaum algebra of the first order Łukasiewicz logic which is not semisimple, because the valid but unprovable formulas are precisely the formulas that correspond to co-infinitesimal elements of the Lindenbaum algebra, see e.g. [7]. Therefore, the study of perfect MV-algebras is tightly connected with this important phenomenon of the first order Łukasiewicz logic.

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In the beginning of the Nineties, two equivalent non-commutative generalizations of MV-algebras, called pseudo MV-algebras in [20] or GMV-algebras in [24], were introduced. They are used for algebraic description of non-commutative fuzzy logic, see [22]. For them the author [11] generalized a well-known Mundici's representation theorem, see e.g. [2, Cor 7.1.8], showing that every pseudo MV-algebra is always an interval in a unital ℓ-group not necessarily Abelian.

From algebraic point of view of perfect MV-algebras, it was shown in [8] that every perfect MV-algebra M can be represented as an interval in the lexicographic product, i.e. $M \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$. This result was extended also for perfect effect algebras [12].

This notion was generalized in [13] to n-perfect pseudo MV-algebras, which can be decomposed into $(n + 1)$-comparable slices, and they can be represented in the form $\Gamma(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, (1, 0))$. $\mathbb{R}$-perfect pseudo MV-algebras can be represented in the form $\Gamma(\mathbb{R} \overrightarrow{\times} G, (1, 0))$, see [14]. If G is Abelian, such MV-algebras were studied in [9]. Recently, lexicographic MV-algebras were studied in [5]. They have a representation in the form $\Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is an Abelian linearly ordered group and G is an Abelian ℓ-group.

Thus we see that MV-algebras and pseudo MV-algebras that can be represented in the form $\Gamma(H \overrightarrow{\times} G, (u, 0))$ are intensively studied in the last period, see [15] where H was assumed to be Abelian. In this contribution, we continue in this study exhibiting the most general case of (H, u) and G when they are not assumed to be Abelian. We show that the crucial conditions are the existence of a lexicographic ideal, or equivalently, the possibility of decomposing M into comparable slices indexed by the elements of the interval $[0, u]_H$; we call such algebras (H, u)-perfect. In addition, we present also conditions when M can be represented in the form $\Gamma(H \overrightarrow{\times} G, (u, b))$, where $b \in G^+$ is not necessarily the zero element.

The paper is organized as follows. The second section gathers the basic notions on pseudo MV-algebras. In the third section we introduce a lexicographic ideal, and we present a representation of a pseudo MV-algebra in the form $\Gamma(H \overrightarrow{\times} G, (u, 0))$. Section 4 gives a categorical equivalence of the category of (H, u)-lexicographic pseudo MV-algebras to the category of ℓ-groups. The final section will describe weakly (H, u)-perfect pseudo MV-algebras; they can be represented in the form $\Gamma(H \overrightarrow{\times} G, (u, b))$, where b can be even strictly positive. Crucial notions for such algebras are a weakly lexicographic ideal as well as a weakly (H, u)-perfect pseudo MV-algebra.

2. Basic notions on pseudo MV-algebras

According to [20], a *pseudo MV-algebra* or a *GMV-algebra* by [24] is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

We note that $\odot$ has a higher binding priority than $\oplus$.

Any pseudo MV-algebra is a distributive lattice where (A6) and (A7) define the join $x \vee y$ and the meet $x \wedge y$ of x, y, respectively.