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Ordered domain algebras

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ABSTRACT

We give a finite axiomatisation to representable ordered domain algebras and show that finite algebras are representable on finite bases.

1. Introduction

Domain algebras provide an elegant, one-sorted formalism for automated reasoning about program and system verification, see [4,3] and [7] for details and further motivation. The algebraic behaviour of domain algebras have been investigated, e.g. in [5,6]. Their primary models are algebras of relations, viz. representable domain algebras. P. Jipsen and G. Struth raised the question whether the class $\mathsf{R}(;, \mathsf{dom})$ of representable domain algebras of the minimal signature $(;, \mathsf{dom})$ is finitely axiomatisable. To formulate the question precisely, let us recall the definition of representable domain algebras $\mathsf{R}(;, \mathsf{dom})$.

Definition 1.1. The class $\mathsf{R}(;, \mathsf{dom})$ is defined as the isomorphs of $\mathcal{A} = (A, ;, \mathsf{dom})$ where $A \subseteq \wp(U \times U)$ for some base set U and

$$x ; y = \{(u, v) \in U \times U\colon (u, w) \in x \text{ and } (w, v) \in y \text{ for some } w \in U\}$$

$$\mathsf{dom}(x) = \{(u, u) \in U \times U\colon (u, v) \in x \text{ for some } v \in U\}$$

for every $x, y \in A$.

The signature $(;, \mathsf{dom})$ can be expanded to larger signatures τ by including other operations. For instance, we can define

$$\mathsf{ran}(x) = \{(v, v) \in U \times U\colon (u, v) \in x \text{ for some } u \in U\}$$

$$x^{\smile} = \{(v, u) \in U \times U\colon (u, v) \in x\}$$

$$1' = \{(u, v) \in U \times U\colon u = v\}$$

and also include the bottom element 0 (interpreted as the empty set $\emptyset$) and the ordering $\leqslant$ (interpreted as the subset relation $\subseteq$) to yield representable algebraic structures. The corresponding representation classes $\mathsf{R}(\tau)$ for larger signatures τ are defined analogously to the definition of $\mathsf{R}(;, \mathsf{dom})$.

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It turned out that the answer to the above problem is negative.

Theorem 1.2. *(See [7].) Let τ be a similarity type such that $(;, \mathsf{dom}) \subseteq \tau \subseteq (;, \mathsf{dom}, \mathsf{ran}, 0, 1')$. The class $\mathsf{R}(\tau)$ of representable τ-algebras is not finitely axiomatisable in first-order logic.*

Note that the above theorem does not apply to signatures where the ordering $\leqslant$ is present. In fact, D.A. Bredikhin proved [2] that the class $\mathbf{R}(;, \mathsf{dom}, \mathsf{ran}, ^\smile, \leqslant)$ of representable algebraic structures is finitely axiomatisable. Bredikhin proves finite axiomatisability by reducing the problem of representing an ordered domain algebra (see the definition below for this axiomatically given class of algebraic structures) to that of an ordered involuted semigroup. Representable (ordered) involuted semigroups were characterised independently by Bredikhin [1] and B.M. Schein [8]. These characterisations use infinitely many quasi-equations (in fact, the class of representable involuted semigroups is a non-finitely axiomatisable quasi-variety [2]) and the representations are rather involved (in the case of Schein's proof, an infinitary construction called "graph-theoretic scaffolding" is used).

In this paper we take a more direct approach in proving finite axiomatisation of representable ordered domain algebras. The advantage of our proof is that it uses a Cayley-type representation of abstract algebraic structures that also shows *finite representability*, i.e. that finite elements of $\mathbf{R}(;, \mathsf{dom}, \mathsf{ran}, ^\smile, 0, 1', \leqslant)$ can be represented on finite bases. In passing we note that if composition is not definable in τ, then $\mathsf{R}(\tau)$ has the finite representation property, but it can be shown that every signature containing $(\cdot, ;, 1')$ or $(\cdot, ;, ^\smile)$ (where $\cdot$ is interpreted as intersection) fails to have the finite representation property.

2. Main result

Let *Ax* denote the following formulas. These axioms are essentially the same as in [2], we just made slight adjustments to include the constants $1'$ and 0 as well.

Partial order: The ordering $\leqslant$ is reflexive, transitive and antisymmetric, with lower bound 0.

Monotonicity and normality: The operations $;, \mathsf{dom}, \mathsf{ran}, ^\smile$ are monotonic, e.g. $a \leqslant b$ implies $a ; c \leqslant b ; c$, etc. and normal $0^\smile = 0 ; a = a ; 0 = \mathsf{dom}(0) = \mathsf{ran}(0) = 0$.

Involuted monoid: The operation ; is associative, the constant $1'$ is a left and right identity for ;, $^\smile$ is an involution $(a^\smile)^\smile = a$ and $(a ; b)^\smile = b^\smile ; a^\smile$, and $1'^\smile = 1'$.

Domain/range axioms:

$$\mathsf{dom}(a) = \big(\mathsf{dom}(a)\big)^\smile \leqslant 1' = \mathsf{dom}\big(1'\big) \tag{1}$$

$$\mathsf{dom}(a) \leqslant a ; a^\smile \tag{2}$$

$$\mathsf{dom}\big(a^\smile\big) = \mathsf{ran}(a) \tag{3}$$

$$\mathsf{dom}\big(\mathsf{dom}(a)\big) = \mathsf{dom}(a) = \mathsf{ran}\big(\mathsf{dom}(a)\big) \tag{4}$$

$$\mathsf{dom}(a) ; a = a \tag{5}$$

$$\mathsf{dom}(a ; b) = \mathsf{dom}\big(a ; \mathsf{dom}(b)\big) \tag{6}$$

$$\mathsf{dom}\big(\mathsf{dom}(a) ; \mathsf{dom}(b)\big) = \mathsf{dom}(a) ; \mathsf{dom}(b) = \mathsf{dom}(b) ; \mathsf{dom}(a) \tag{7}$$

A model $\mathcal{A} = (A, ;, \mathsf{dom}, \mathsf{ran}, ^\smile, 0, 1', \leqslant)$ of these axioms is called an *ordered domain algebra*, and the class of ordered domain algebras is denoted by ODA.

Each of the axioms (1)–(7) has a dual axiom, obtained by swapping domain and range and reversing the order of compositions. We denote the dual axiom by a ∂ superscript, thus for example, $(6)^\partial$ is $\mathsf{ran}(b ; a) = \mathsf{ran}(\mathsf{ran}(b) ; a)$. The dual axioms can be obtained from the axioms above, using the involution axioms and (3).

The operations can be easily extended to subsets of elements as follows. Let $\mathcal{A} \in \mathsf{ODA}$ and $X, Y \subseteq A$. We define

$$X^\smile = \big\{x^\smile : x \in X\big\}$$

$$X ; Y = \{x ; y : x \in X,\ y \in Y\}$$

$$\mathsf{dom}(X) = \big\{\mathsf{dom}(x) : x \in X\big\}$$

$$\mathsf{ran}(X) = \big\{\mathsf{ran}(x) : x \in X\big\}$$

Note that we do not claim that the algebra of subsets satisfy the axioms.

We will need the following notation

$$X^\uparrow = \{a \in A : a \geqslant x \text{ for some } x \in X\}$$