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Representable posets

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ABSTRACT

A poset is representable if it can be embedded in a field of sets in such a way that existing finite meets and joins become intersections and unions respectively (we say finite meets and joins are preserved). More generally, for cardinals α and β a poset is said to be (α, β)-representable if an embedding into a field of sets exists that preserves meets of sets smaller than α and joins of sets smaller than β. We show using an ultraproduct/ultraroot argument that when $2 \leq \alpha, \beta \leq \omega$ the class of (α, β)-representable posets is elementary, but does not have a finite axiomatization in the case where either α or $\beta = \omega$. We also show that the classes of posets with representations preserving either countable or *all* meets and joins are pseudoelementary.

1. Introduction

Every poset P can be thought of as a set of sets ordered by inclusion by considering the embedding $P \to \wp(P)$ defined by $p \mapsto \{q \in P : q \leq p\}$. Indeed, this representation has the advantage of preserving all existing meets, finite and infinite. Similarly the representation defined by $p \mapsto \{q \in P : q \ngeq p\}$ preserves existing joins [1]. It follows from this that semilattices can be represented in such a way that their binary operation is modelled by union or intersection appropriately, though in general we cannot construct representations where both existing joins and meets are interpreted as unions and intersections respectively.

Distributivity, or the lack of it, is the issue here, as might be expected given that fields of sets are distributive. In the case of Boolean algebras, which are necessarily distributive, representability follows as an easy corollary of Stone's theorem [14]. For lattices distributivity alone is a necessary and sufficient condition for representability.

Definition 1.1 *(Lattice representation).* Let L be a bounded lattice. A *representation* of L is a bounded lattice embedding $h\colon L \to \wp(X)$ for some set X, where $\wp(X)$ is considered as a field of sets, under the operations of set union and intersection. When such a representation exists we say that L is *representable*.

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Theorem 1.2 *(Birkhoff). Let L be a distributive lattice, let F be a filter of L, and let I be an ideal of L with $F \cap I = \emptyset$. Then there is a prime filter $F' \subset L$ with $F \subseteq F'$ and $I \cap F' = \emptyset$.*

As a corollary to this we have the following result.

Theorem 1.3. *A bounded lattice is representable if and only if it is distributive.*

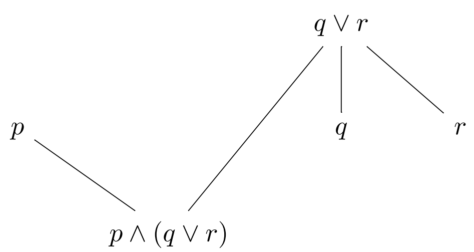
Proof. If L is a distributive lattice and K is the set of prime filters of L it's not difficult to show that the map $h\colon P \to \wp(K)$ defined by $h(a) = \{f \in K : a \in f\}$ is a representation using Theorem 1.2. The converse is trivial. □

In the meet-semilattice case we have representability for existing finite joins if and only if an infinite family of first order axioms demanding that whenever $a \wedge (b_1 \vee \ldots \vee b_n)$ is defined, then $(a \wedge b_1) \vee \ldots \vee (a \wedge b_n)$ is also defined and the two are equal is satisfied [3,11]. In the lattice case of course only a single distributivity equation is both necessary and sufficient. It is possible that in the meet-semilattice case meets could distribute over some finite cardinalities of existing joins but not others, which would necessitate a family of axioms to axiomatize representability. Schein [11] asserts that this is the case, and indeed that an infinite family is required, but there is some reason to doubt that this has been established as a fact. We expand on this in the discussion following Corollary 2.9.

It is tempting to try to apply the meet-semilattice axiom schema directly to posets. There is a problem however, as the classical Zorn's lemma based argument for semilattices and lattices does not work in the poset case. A step in this argument assumes the existence of finite meets, which is invalid in the more general setting of posets. This is a symptom of a deeper problem, as a simple generalization of the axiom schema for semilattices (such as appears as the condition LMD in [5], see Definition 1.4 below) is not expressive enough for the poset situation (see Example 1.5, where it is shown that LMD is not a necessary condition for a poset to have a representation). Note that the question of whether LMD is a *sufficient* condition for representability with respect to all meets and binary joins is raised as open in [5], and appears not to have been resolved.

Definition 1.4 *(LMD).* A poset P is *LMD* if and only if, for all $x, y, z \in P$, whenever $x \wedge (y \vee z)$ is defined then $(x \wedge y) \vee (x \wedge z)$ is also defined and they are equal.

Example 1.5 *(The condition **LMD** is not necessary for representability).* Let P be the poset



Then $p \wedge q$, $p \wedge r$, and $(p \wedge q) \vee (p \wedge r)$ do not exist in P but P is representable. For example using the field of sets generated by $\{a, c, d\}, \{b, d\}, \{b, c\}$, where $p \mapsto \{a, c, d\}$, $q \mapsto \{b, d\}$ and $r \mapsto \{b, c\}$.

Suppose we were to weaken *LMD* so that equality is only necessary in the case where both sides are defined (see Definition 1.6 below)? It turns out that an axiom schema along these lines is not sufficient to ensure representability. We will demonstrate this in Example 2.13 later.