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A logic of non-monotonic interactions

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ABSTRACT

In this paper, which is part of the Zsyntax project outlined in Boniolo et al. (2010) [2], we provide a proof-theoretical setting for the study of context-sensitive interactions by means of a non-monotonic conjunction operator. The resulting system is a non-associative variant of MLL_{pol} (the multiplicative polarised fragment of Linear Logic) in which the monotonicity of interactions, depending on the context, is governed by specific devices called *control sets*. Following the spirit of Linear Logic, the ordinary sequent calculus presentation is also framed into a theory of proof-nets and the set of sequential proofs is shown to be sound and complete with respect to the class of corresponding proof-nets. Some possible biochemical applications are also discussed.

1. Introduction

Jane has a strong preference for red wine over white and for white wine over beer. She also has a weak preference for fish over beef. If presented with a menu that contains *beef* (*bf*), *fish* (*fs*), *white wine* (*ww*) and *beer* (*br*), she will probably choose according to her best available choice in each category. Schematically:

$$fs \otimes ww \otimes bf \otimes br \vdash (fs \odot ww) \otimes bf \otimes br \tag{1}$$

where (i) $\otimes$ is a resource-sensitive conjunction, indicating that the conjoined items are simultaneously available for ordering, each operand representing a single consumable item of the given type (so that the number of times the name of an item-type occurs in the $\otimes$-expression reflects the number of consumable items of that type that are available),[1] (ii) "$\odot$" denotes a "meal composition" operator, and (iii) the relation symbol "$\vdash$" denotes the pseudo-consequence relation that obtains between states A and B when B is reachable from A by a (possibly empty) sequence of choosing acts (so that, for every A, it holds that $A \vdash A$ as a result of the empty sequence of choosing acts). Hence, in the above example we are assuming that exactly one item of each dish and drink is available. In the right-hand side, as a result of Jane's choosing act, the units of fish and white wine have been used up to fulfil Jane's order, while the others remain available for further consumption.

What happens if one unit of red wine (*rw*) is added to the menu? It may well be that Jane chooses to order beef and red wine despite her weak preference for fish over beef, because to her taste red wine (for which she has a strong preference) "binds better" to beef than to fish. That is, her choice in the new extended menu is represented by

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[1] Here we make the simplifying assumption that the order in which the menu items are presented is immaterial, while it is well-known that it may well affect one's choice. However the approach developed in the following sections is largely independent of this simplifying assumption.

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$$fs \otimes ww \otimes bf \otimes br \otimes rw \vdash (bf \odot rw) \otimes fs \otimes ww \otimes br. \tag{2}$$

On the other hand, given that Jane dislikes cider (cd), its addition to the original menu in (1), instead of red wine, would by no means alter her initial choice:

$$fs \otimes ww \otimes bf \otimes br \otimes cd \vdash (fs \odot ww) \otimes bf \otimes br \otimes cd \tag{3}$$

and the same happens if the menu is extended with any item (e.g., chicken, tea, etc.) other than red wine.

Now, observing (1), (3) and the like, the $\otimes$ operator appears to be *monotonic*, i.e., to satisfy the following condition:

$$\frac{A \vdash C}{A \otimes B \vdash C \otimes B} \tag{4}$$

where A, B, C are arbitrary expressions representing states of the process, constructed using the resource-sensitive operators $\otimes$ and $\odot$. However, observing (1) and (2), it turns out to be *non-monotonic* and violate (4).

If we insist that an operator $\otimes$ should be classified as "monotonic" only if it *always* satisfies (4), then we should conclude that our $\otimes$ operator is non-monotonic. However, this involves some loss of information. For instance, on the basis of Jane's preferences we may *know in advance* that she will always make the same order as in (2) when presented with any menu that contains red wine and beef, quite independent of whatever else is available, that is:

$$bf \otimes rw \otimes A \vdash (bf \odot rw) \otimes A \tag{5}$$

whatever A may be. Moreover we may also know, still on the basis of her preferences, that she will always order fish and white wine *unless* beef and red wine are both present in the menu. This kind of information may be schematically represented as follows:

$$fs \otimes ww \otimes A \vdash (fs \odot ww) \otimes A \quad \text{provided that} \quad rw \text{ and } bf \text{ are not in } A. \tag{6}$$

Observe that the kind of information displayed in (5) and (6) can be obtained and used without knowing all the details of Jane's preferences.

The main purpose of this paper is to envisage a formalism for adequately representing, without loss of information, processes that involve this kind of *context-sensitive interactions*, which prompt for a *controlled monotonicity* of the $\otimes$ operator. We maintain that this approach may turn out to be useful in a variety of applications. A prominent one is the representation of biochemical processes where this kind of controlled monotonicity allows for expressing the empirical regularities that are typically observed in the laboratory, which often involve context-sensitive, non-monotonic interactions between objects.

An example is given by the all-important mechanisms concerning concurrent enzyme inhibition. An enzyme inhibitor is a molecule that binds to enzymes and decreases their activity. Since blocking an enzyme's activity can, for example, kill a pathogen or correct a metabolic imbalance, many drugs are enzyme inhibitors. Let us indicate by E the enzyme and by S its substratum, that is the molecule that should bind to the enzyme. Then, in most contexts, we have that $E \otimes S \vdash E \odot S$. However, if the inhibitor I is present, we observe that $E \otimes S \otimes I \vdash (E \odot I) \otimes S$ and $E \otimes S \otimes I \nvdash (E \odot S) \otimes I$, violating (4). Since most biological "regularities" are strongly context-dependent, the standard monotonicity of $\otimes$ cannot hold in general. However, simply saying that it does not hold would hide the important information that certain reactions may be assumed to take place in virtually all contexts, *except for* some known ones in which another kind of reaction is observed. With respect to this class of applications, the present paper can be seen as a development of the *Z-Syntax* project outlined in [2].

Following a tradition started with the Curry–Howard correspondence we shall present our formalism as a *logical* one, in which formulas are interpreted as *types* of objects and proofs are interpreted as *processes*. Under this interpretation, as in the examples above, the relation $\vdash$ is informally explained as follows:

$A \vdash B$ ("B is reachable from A") iff there exists a process (possibly the null one) that,

starting from an aggregate of type A, yields an aggregate of type B. (7)

Such processes typically involve (context-sensitive) non-monotonic interactions of objects that are destroyed to generate new ones. So, this approach requires that once a formula has been used in a proof, it is no longer available for further processing unless a fresh copy of it is provided, that is, the classical (and intuitionistic) *contraction* rule:

$$\frac{A \otimes A \vdash B}{A \vdash B} \tag{Contraction}$$

is *not* valid. Given the general failure of contraction, the logic we are concerned with belongs to the family of *substructural logics* [17,13,4,12,15,1,16], but departs from it for the important fact that (4), which is valid in all known substructural logics, is *not* valid in general.

As customary, we can identify an expression $A_1 \otimes \cdots \otimes A_n$, when occurring on the left of $\vdash$, with the *multiset* $[A_1, \ldots, A_n]$ and often represent such multisets by simply listing their elements (in an arbitrary order). Then, the monotonicity of $\otimes$ can be expressed by