Journal of Applied Logic 12 (2014) 377–394

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

Propositional dynamic logic for searching games with errors

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ARTICLE INFO

Article history:
Received 4 February 2013
Accepted 7 April 2014
Available online 13 April 2014

Keywords:
Propositional dynamic logic
Many-valued logics
Łukasiewicz logic
Kripke models
MV-algebras

ABSTRACT

We investigate some finitely-valued generalizations of propositional dynamic logic with tests. We start by introducing the $n+1$-valued Kripke models and a corresponding language based on a modal extension of Łukasiewicz many-valued logic. We illustrate the definitions by providing a framework for an analysis of the Rényi–Ulam searching game with errors.

Our main result is the axiomatization of the theory of the $n+1$-valued Kripke models. This result is obtained through filtration of the canonical model of the smallest $n+1$-valued propositional dynamic logic.

1. Introduction

Propositional dynamic logic (PDL) is a multi-modal logic designed to reason about programs. The general idea behind the semantic of this system is the following. Program states are gathered in a set W. Any program α is encoded by its input/output relation on W. Programs are built from atomic ones and test operators using regular operations. Their associated relations are defined as to respect this algebra of programs. The goal is to provide a framework for formal verification through input/output specifications.

Since its introduction by Fischer and Ladner in [9], the scope of dynamic logic has widened to many other areas such as game theory (see [13,18,27]), epistemic logic (see [32]) or natural language (see [31]). The subject is under constant and active development (see [1,2,8,19] for example) and we refer to [17] for an introductory monograph.

Informally, PDL is a mixture of modal logic and algebra of regular programs. Recently, some authors have considered generalizations of modal logics to many-valued realms (see [3,4,11,12,10,24]). These modal many-valued systems can naturally be considered as building blocks of many-valued generalizations of PDL. In other words, these developments raise the issue of describing the systems obtained by adding a many-valued flavor to the modal logic used to define PDL. Such many-valued propositional dynamic logics

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[1] The author was supported by the internal research project F1R-MTHPUL-12RDO2 of the University of Luxembourg.

http://dx.doi.org/10.1016/j.jal.2014.04.001
1570-8683/

would provide a language to state correctness criteria in the form of input/output specifications that could be *partly* satisfied.

We address this problem for the modal extensions of the $n+1$-valued Łukasiewicz logics (see [20–22]) studied in [15,16]. Hence, the truth values of the propositions range in a set of finite cardinality $n+1$ where $n \geq 1$.

Our starting point is the definition of a language (with test operator) for such generalizations and their corresponding $n+1$-valued Kripke models. In these models, relations associated to programs are crisp and valuation maps are many-valued. As an illustration of the new possibilities allowed by this language, we explain how it can be used to construct a dynamic model for formal verification of strategies of the Rényi–Ulam searching game with errors.

The goal of this paper is the characterization of the theory of these $n+1$-valued Kripke models (*i.e.*, the set of formulas that are true in any model). In this view, Theorem 5.13 is our main result. It gives an axiomatization of this theory through an $n+1$-valued propositional dynamic deductive system that we denote by PDL_n.

This result is obtained by the way of the canonical model. This construction defects to be a 'standard' $n+1$-valued Kripke model and we need a filtration result to obtain Theorem 5.13.

The construction of the canonical model for PDL is algebraic in disguise. This model is built upon the set of the maximal filters of the Lindenbaum–Tarski algebra of PDL which is a multi-modal Boolean algebra. Naturally, the canonical model for PDL_n also has an algebraic flavor. The system PDL_n is based on modal extensions of Łukasiewicz $n+1$-valued logic. Hence, MV-algebras—which are the algebraic counterpart of Łukasiewicz logics—replace Boolean algebras in this setting.

The techniques used in the proofs in this paper are generalizations of the corresponding techniques for PDL. It is worth noting that by considering $n = 1$, our results boil down to the existing ones for PDL.

This paper is organized as follows. In the next section we introduce some many-valued generalizations of the language and models of PDL. Section 3 provides an example that illustrates the possibilities offered by these generalizations. Section 4 is devoted to the development of a sound deductive system PDL_n for the $n+1$-valued Kripke models. The many-valued forms of the intrinsic axioms of PDL, such as the induction axiom, are discussed when needed. Eventually, in Section 5 we prove the deductive completeness of PDL_n with respect to the $n+1$-valued Kripke-models (proof of the Filtration Lemma is provided in Appendix A). In order to keep the paper self-contained, we recall the necessary definitions and results about algebras of regular programs and MV-algebras.

2. Many-valued Kripke models for dynamic logics

The starting point of the developments of this paper is a generalization to an $n+1$-valued realm of the definitions of the propositional dynamic language and the Kripke models.

Let us denote by Π_0 a nonempty set of atomic programs (denoted by $a, b, \ldots$) and by Prop a countable set of propositional variables (denoted by $p, q, \ldots$). The sets Π of programs and Form of well formed formulas are given by the following Backus–Naur forms (where ϕ are formulas and α are programs):

$$\begin{aligned} \phi &::= p \mid 0 \mid \neg\phi \mid \phi \to \phi \mid [\alpha]\phi \\ \alpha &::= a \mid \phi? \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^*. \end{aligned} \tag{2.1}$$

To extend the definition of a Kripke model to a $[0,1]$-valued realm, we use Łukasiewicz interpretation $\to^{[0,1]}$ and $\neg^{[0,1]}$ of the binary connector $\to$ and the unary connector $\neg$ respectively. These maps are defined on $[0,1]$ by

$$\neg^{[0,1]}x = 1 - x \quad \text{and} \quad x \to^{[0,1]} y = \min(1 - x + y, 1). \tag{2.2}$$