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Second order inductive logic and Wilmers' principle

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ABSTRACT

We extend the framework of Inductive Logic to Second Order languages and introduce Wilmers' Principle, a rational principle for probability functions on Second Order languages. We derive a representation theorem for functions satisfying this principle and investigate its relationship with the first order principles of Regularity and Super Regularity.

1. Introduction

In the framework of Pure Inductive Logic, a rational agent's belief function is usually regarded as a probability function on the set of first order sentences of a certain, fixed language L. This language contains constants representing the objects of the universe and predicates representing the properties of these objects. This allows an agent to express statements about the universe.

So far such statements have, to our knowledge, been limited to first order expressions, allowing the agent to make existential or universal statements about the objects. As the so-called Geach–Kaplan statement[2] shows (see e.g. [1]), an agent could increase her expressive power if she were to extend the set of expressions available to include second order statements.

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[2] This statement is usually given in the form

$$\exists X\big(\exists x \exists y\big(X(x) \wedge X(y) \wedge R(x,y)\big) \wedge \exists x \neg X(x) \wedge \forall x \forall y\big(X(x) \wedge R(x,y) \rightarrow X(y)\big)\big)$$

as an example of a second order statement that cannot be formulated in first order logic. As far as the work presented in this paper is concerned the statement is mentioned merely as a motivational example of why one might want to study second order logic in the first place.

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As we identify an agent's belief in a statement with the value the agent's belief function assigns to it, allowing the agent to use second order expressions will require her to extend the domain of her belief function to include second order sentences.

Unsurprisingly, this leads to a number of complications. Since Second Order logic is inherently incomplete (see e.g. [8]), we will have to be careful picking a suitable framework for Second Order Inductive Logic. At the same time, we would want to have a suitable interpretation of universal and existential quantification over the predicates in L.

In this paper we intend to provide such a framework, allowing an agent to extend her expressive power to second order logic. Once such a framework is given, we can study rational principles involving second order logic. We will give an example of one such principle, called Wilmers' Principle, and provide a representation theorem for second order belief functions that satisfy this principle. We will then consider the consequences of this principle for the thorny question of Universal Truth for first order statements.

2. Second Order probability functions

As with traditional Inductive Logic[3] for example [2], we will work in a unary language L with predicate symbols P_i and constants a_i for $i \in \{1, 2, 3, \dots\} = \mathbb{N}^+$ but without function symbols or equality. Let $F_1 L, S_1 L, QFS_1 L$ respectively denote the first order formulae, sentences and quantifier free sentences of L.

Let $\mathcal{T}L$ denote the set of structures M for L in which the constants a_i are interpreted as themselves and $|M| = \{a_i \mid i \in \mathbb{N}^+\}$, so every element of the universe of M is denoted by a constant symbol. Similarly we shall use P_j to denote $\{a_i \mid M \models P_j(a_i)\}$, leaving the M implicit whenever this cannot cause confusion.

We say that $w : S_1 L \to [0, 1]$ is a *probability function on* $S_1 L$, if for any $\vartheta, \varphi \in S_1 L$, $\psi(x) \in F_1 L$,

(P1) If $\models \vartheta$, then $w(\vartheta) = 1$.
(P2) If $\vartheta \models \neg\varphi$, then $w(\vartheta \vee \varphi) = w(\vartheta) + w(\varphi)$.
(P3) $w(\exists x \psi(x)) = \lim_{n\to\infty} w(\bigvee_{i=1}^n \psi(a_i))$.

To our mind the central problem of (Pure) Inductive Logic can be picturesquely captured as follows: Imagine an agent inhabiting a structure $M \in \mathcal{T}L$ but having no further knowledge, so in particular the agent has no particular interpretation in mind for the constant and predicate symbols. In that case what probability $w(\vartheta)$ should the agent rationally, or logically, give to $\vartheta \in S_1 L$? Or more precisely, since we obviously intend for these probability values to be coherent, what probability function w should the agent rationally or logically adopt?

In the absence of any clear definition of what is meant here by 'rationally' (which for the purpose of this paper we identify with 'logically') the usual method of tackling this question is by imposing certain ostensibly rational, or at least not irrational, requirements on w and seeing where that leads. For example the symmetry between the constants a_i, and between the predicates P_j, from the agent's point of view surely requires that w should satisfy:

Constant Exchangeability, Ex. *w satisfies Ex, if for all $\vartheta \in S_1 L$ and all permutations σ of $\mathbb{N}^+$,*

$$w\big(\vartheta(a_1, \dots, a_n)\big) = w\big(\vartheta(a_{\sigma(1)}, \dots, a_{\sigma(n)})\big).$$

Predicate Exchangeability, Px. *w satisfies Px, if for all $\vartheta \in S_1 L$, and all permutations σ of $\mathbb{N}^+$,*

[3] Actually Inductive Logic is more commonly presented with only finitely many predicate symbols but as we would in any case advocate the rationality of Unary Language Invariance in this context, see for example [7, Chapter 16], this would ultimately lead to the same situation.