Journal of Applied Logic 12 (2014) 558–569

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

A logical calculus for controlled monotonicity

Marcello D'Agostino [a,*], Mario Piazza [b], Gabriele Pulcini [c,1]

[a] *Dipartimento di Economia e Management, Università di Ferrara, Italy*
[b] *Dipartimento di Scienze Filosofiche, Pedagogiche ed Economico-Quantitative, Università di Chieti–Pescara, Italy*
[c] *Department of Philosophy & Center for Logic, Epistemology and History of Science, State University of Campinas, Brazil*

ARTICLE INFO

Article history:
Received 17 March 2014
Received in revised form 11 June 2014
Available online 27 August 2014

Keywords:
Substructural logics
Non-monotonicity
Cut-elimination
Series-parallel structures

ABSTRACT

In this paper we introduce a new deductive framework for analyzing processes displaying a kind of *controlled* monotonicity. In particular, we prove the cut-elimination theorem for a calculus involving series-parallel structures over partial orders which is built up from *multi-level* sequents, an interesting variant of Gentzen-style sequents. More broadly, our purpose is to provide a general, syntactical tool for grasping the combinatorics of non-monotonic processes.

1. Introduction

In structural proof-theory, the monotonicity of classical and intuitionistic logic is standardly expressed by the fact that new incoming information does not affect the derivability of previously obtained conclusions. In Gentzen-style sequent calculi, this sort of informative insularity is usually guaranteed by means of the left-weakening rule:

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}$$

Yet, what has been neglected by logicians is a subtler kind of monotonicity holding at the level of *connectives*. Consider the right-conjunction rule:

* Corresponding author.
E-mail addresses: marcello.dagostino@unife.it (M. D'Agostino), m.piazza@unich.it (M. Piazza), gab.pulcini@gmail.com (G. Pulcini).
[1] The author acknowledges support from FAPESP Post-Doc Grant 2013/22371-0, São Paulo State, Brazil.

http://dx.doi.org/10.1016/j.jal.2014.08.001
1570-8683/

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \vdots & \vdots\\ \Gamma \vdash A & \Delta \vdash B\end{array}}{\Gamma, \Delta \vdash A \wedge B}$$

The contexts Γ and Δ, when glued together, produce A and B, nevertheless a kind of monotonicity is at work since Δ (resp. Γ) does not inhibit the deduction of A (resp. B). Clearly, this kind of monotonicity still holds when weakening and contraction rules are dropped from classical or intuitionistic sequent calculus, insofar as the above derivation remains valid in linear logic with the multiplicative conjunction $\otimes$ in place of $\wedge$. This leads us to a general definition of *monotonic* conjunction. A conjunction $\star$ is called *monotonic* in a logic $\mathscr{L}$ when, for all formulas A, B and C, both the following rules are admissible in $\mathscr{L}$:

$$\frac{A \vdash B}{A \star C \vdash B \star C} \qquad\qquad \frac{A \vdash B}{C \star A \vdash C \star B}$$

Although this definition can be easily generalized to the other binary operators, in this paper we will focus on conjunctions. In [2] this definition serves as an antechamber for the introduction of the substructural sequent calculus CSI. The logical platform of CSI is that of multiplicative linear logic [6]. In this sense, this calculus is not the first attempt to encode non-monotonic patterns of inference by means of linear logic [4,3]. However, CSI is a first proposal for representing processes involving *context-sensitive interactions* (so the acronym), which prompt for a *controlled monotonicity* of the $\otimes$-conjunction. Indeed, in CSI the $\otimes$-conjunction is split into two versions corresponding to its monotonic and non-monotonic side:

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \vdots & \vdots\\ (\Gamma) \vdash A & (\Delta) \vdash B\end{array}}{(\Gamma, \Delta) \vdash A \otimes B}\otimes_{\mathcal{R}}(1) \qquad \dfrac{\begin{array}{cc}\pi_1 & \pi_2\\ \vdots & \vdots\\ (\Gamma) \vdash A & (\Delta) \vdash B\end{array}}{(\Gamma), (\Delta) \vdash A \otimes B}\otimes_{\mathcal{R}}(2)$$

The sequent $(\Gamma, \Delta) \vdash A \otimes B$ expresses the fact that the conclusion $A \otimes B$ is attainable once the resources in Γ and in Δ are put together at the same time, in the same environment. Differently, $(\Gamma), (\Delta) \vdash A \otimes B$ affirms that the same formula $A \otimes B$ can be obtained from Γ and Δ keeping the two clusters of resources separate, i.e. same time, but different environments. Thereby, the meaning of these rules is the following: if something in Δ (resp. Γ) inhibits A (resp. B), then the second rule can be applied, but not the first:

$$(\Gamma), (\Delta) \vdash A \otimes B \quad \text{but} \quad (\Gamma, \Delta) \nvdash A \otimes B$$

Let us a look at a biochemical example. Consider the enzyme E that binds with the substratum S (in symbols $E \odot S$) only in absence of the inhibitor I, otherwise the compound produced is $E \odot I$. This fact can be expressed by saying that the sequent $\Gamma, E, S \vdash \bigotimes \Gamma \otimes (E \odot S)$ is valid provided that $I \notin \Gamma$. In other words, we have that:

$$\dfrac{\dfrac{\dfrac{(E) \vdash E \qquad (S) \vdash S}{(E, S) \vdash ES}\otimes_{\mathcal{R}}(1)\text{ monotonic!}}{(E, S) \vdash E \odot S}\odot \qquad (I) \vdash I}{(E, S), (I) \vdash (E \odot S) \otimes I}\otimes_{\mathcal{R}}(2)\text{ non-monotonic!}$$

The same proof can be turned into an unsound derivation once that the $\otimes_{\mathcal{R}}(2)$-rules is replaced by its monotonic version $\otimes_{\mathcal{R}}(1)$.