Journal of Applied Logic 12 (2014) 570–583

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

Products of modal logics and tensor products of modal algebras

Dov Gabbay [a,b,c,*], Ilya Shapirovsky [d,*], Valentin Shehtman [d,e,*]

[a] *Department of Computer Science, Bar Ilan University, Israel*
[b] *Department of Informatics, King's College London, United Kingdom*
[c] *Faculty of Science, Technology and Communication, University of Luxembourg, Luxembourg*
[d] *Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia*
[e] *National Research University Higher School of Economics, Moscow, Russia*

A R T I C L E I N F O

Article history:
Received 10 July 2013
Accepted 21 August 2014
Available online 27 August 2014

Keywords:
Product of modal logics
Tensor product
Logical invariance
Modal algebra
Finite model property
Filtration
Tabular logic

A B S T R A C T

One of natural combinations of Kripke complete modal logics is the product, an operation that has been extensively investigated over the last 15 years. In this paper we consider its analogue for arbitrary modal logics: to this end, we use product-like constructions on general frames and modal algebras. This operation was first introduced by Y. Hasimoto in 2000; however, his paper remained unnoticed until recently. In the present paper we quote some important Hasimoto's results, and reconstruct the product operation in an algebraic setting: the Boolean part of the resulting modal algebra is exactly the tensor product of original algebras (regarded as Boolean rings). Also, we propose a filtration technique for Kripke models based on tensor products and obtain some decidability results.

1. Introduction

Products were introduced in the 1970s as a natural type of combined modal logics. They arise in different areas of pure and applied logic – spatial reasoning, multi-agent systems, quantified modal and intuitionistic logics etc. The theory of products was systematized and essentially developed first in the paper [3] and later in the monograph [4]; during the past 10 years new important results were proved and the research is going on, cf. [7].

Recall that the product of modal logics $\mathrm{L}_1, \mathrm{L}_2$ is defined as the logic of the class of products of their Kripke frames

$$\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{Log}\big(\{\mathsf{F}_1 \times \mathsf{F}_2 \mid \mathsf{F}_1 \vDash \mathrm{L}_1,\ \mathsf{F}_2 \vDash \mathrm{L}_2\}\big),$$

and the frame $\mathsf{F}_1 \times \mathsf{F}_2$ inherits the horizontal relations from F_1 and the vertical relations from F_2.

On the one hand, this definition is quite natural, and in some cases products can be simply axiomatized and have nice properties.

* Corresponding authors.

http://dx.doi.org/10.1016/j.jal.2014.08.002
1570-8683/

On the other hand, the product operation has some peculiarities.

First, products are always Kripke complete. However, Kripke semantics sometimes may be inadequate. This means that different logics $\mathrm{L}_1, \mathrm{L}_1'$ can have the same frames; in that case $\mathrm{L}_1 \times \mathrm{L}_2 = \mathrm{L}_1' \times \mathrm{L}_2$ for any L_2 – which looks strange.

Second, the product operation is not logically invariant: it may happen that for some frames $\mathrm{Log}(\mathsf{F}) = \mathrm{Log}(\mathsf{F}')$, while $\mathrm{Log}(\mathsf{F} \times \mathsf{G}) \neq \mathrm{Log}(\mathsf{F}' \times \mathsf{G})$. Note that on the contrary, logical invariance holds for direct products of classical models: $Th(M) = Th(M')$ implies $Th(M \times N) = Th(M' \times N)$ (A. Mostowski, 1952; cf. [8, Theorem 19]).

Third, the product of consistent logics L_1 and L_2 may be inconsistent. This happens if L_1 does not have frames (which is possible in the polymodal case, cf. [10], [5, p. 55]).

To amend the situation, one can try to define products of general frames or, equivalently, modal algebras. The following problem was mentioned in [7, p. 877]:

There are several attempts for extending the product construction from Kripke complete logics to arbitrary modal logics, mainly by considering product-like constructions on Kripke models. All the suggested methods so far result in sets of formulas that are not closed under the rule of Substitution.

Nevertheless, a possible answer was already known by that time: it was given by Y. Hasimoto who introduced so called *shifted products* of general frames and modal algebras [6]. In the present paper we revisit this operation. We call it the *tensor product*, since it acts exactly as tensor multiplication on the Boolean parts of modal algebras (regarded as Boolean rings). These shifted (tensor) products are known to be logically invariant and enjoy other nice properties [6].

The paper is organized as follows. Section 2 contains some basic definitions from modal logic. The correlation between modal products of frames and tensor products of Boolean algebras is established in Section 3. Section 4 recalls some basic properties of tensor products stated in [6]. In Section 5 we propose a filtration technique for Kripke models over tensor products and obtain some decidability results.

2. Preliminaries

We assume that the reader is familiar with basic notions in modal logic (see e.g. [1,4]). We recall some of them, mainly for the sake of notation.

An *n-modal algebra* is a Boolean algebra with additional unary operations $\Diamond_1, \ldots, \Diamond_n$ (*modalities*) such that $\Diamond_i 0 = 0$ and $\Diamond_i(x \vee y) = \Diamond_i x \vee \Diamond_i y$ for all i (if $n = 1$, we omit the subscript '1'). Fix a countable set of propositional variables $PV = \{p_1, p_2, \ldots\}$; ML_n denotes the set of all *n-modal formulas*, i.e., terms over PV in the signature of n-modal algebras.

The notation $A \vDash \varphi$ means that a formula φ is *valid in* an algebra A, i.e., $\varphi = 1$ is true in A under any assignment of propositional variables; φ is *valid in a class* of algebras $\mathfrak{A}$ (in symbols, $\mathfrak{A} \vDash \varphi$) if it is valid in every algebra from $\mathfrak{A}$. The set of all formulas valid in an algebra A is called *the logic of* A and denoted by $\mathrm{Log}(A)$. For a set of formulas Ψ, a *Ψ-algebra* is an algebra A validating all formulas from Ψ (in symbols, $A \vDash \Psi$).

Normal propositional n-modal logics can be defined syntactically or alternatively, as logics of n-modal algebras [1].

A *Kripke n-frame* is a tuple $\mathsf{F} = (W, R_1, \ldots, R_n)$, where R_i are binary relations on a nonempty set W. The *modal algebra* of F (denoted by $MA(\mathsf{F})$) is obtained from the Boolean algebra 2^W of all subsets of a set W by expansion with the operations R_i^{-1}, $i = 1, \ldots, n$ such that for any set $U \subseteq W$, $R_i^{-1}(U) := \{y \mid \exists x \in U\ yR_i x\}$; cf. [1]. The *logic of* F (in symbols, $\mathrm{Log}(\mathsf{F})$) can be defined as $\mathrm{Log}(MA(\mathsf{F}))$.

A general n-frame is a tuple $\mathsf{F} = (W, R_1, \ldots, R_n, A)$, where $(W, R_1, \ldots, R_n)$ is a Kripke frame and A is a subalgebra of $MA(W, R_1, \ldots, R_n)$. The logic of A is also called *the logic of* F and denoted by $\mathrm{Log}(\mathsf{F})$. *A valuation in* F is a valuation in A, i.e., a map $PV \longrightarrow A$.