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Continuity and geometric logic

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ABSTRACT

This paper is largely a review of known results about various aspects of geometric logic. Following Grothendieck's view of toposes as generalized spaces, one can take geometric morphisms as generalized continuous maps. The constructivist constraints of geometric logic guarantee the continuity of maps constructed, and can do so from two different points of view: for maps as point transformers and maps as bundles.

1. Introduction

Geometric logic has arisen in topos theory out of the fact that toposes (by "topos" unqualified we shall always mean Grothendieck topos) may be described as classifying toposes for geometric theories – that is to say, any topos may be presented as being generated by a generic model of some geometric theory.

The historical roots of this idea must surely go back to Grothendieck's dictum that "A topos is a generalized topological space" [10], but there is a specific technical understanding that underlies this: that for the purposes of sheaf cohomology, what was important was particular categorical structure and properties of categories of sheaves over spaces; and that it was fruitful to generalize to other categories (the toposes) with the same structure and properties.

I have not been able to trace in detail how this developed over the 1970s into the idea of toposes as geometric theories as mentioned above. Some of the difficulties are described in the 1986 paper *Theories as Categories* [8], which grew out of notes I made on a talk given by Mike Fourman to computer scientists and gave in outline form some of the ideas and results on which the present paper is based. Fourman said,

> This theory and its applications developed initially without the benefit of widespread publication. Many ideas were spread among a relatively small group, largely by word of mouth. The result of this is that the literature does not provide an accessible introduction to the subject. ... To apportion credit for the ideas presented here is difficult so long after the event. Lawvere and Joyal have a special position in this subject. Many others ... contributed also. Their contributions are, in general, better reflected in their published work.

The difficulty is compounded by the fact that for a long time the terminology was not settled. In both [13] and [20], standard references for topos theory, "geometric" is used for the coherent fragment, without infinitary disjunctions, and classifying toposes are constructed for coherent theories. By contrast [23], which also sets out the results on classifying toposes, uses "coherent" (or $L^{g}_{\infty\omega}$) for the full geometric logic, and "finitary coherent" for what we call coherent. [17, D1.1.6] uses the terms as we have them here.

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I would also refer the reader to [31], which shows in more detail how results in the standard texts [20,16,17] justify the view of toposes described here.

While geometric logic can be treated as just another logic, it is an unusual one. Much of this arises from its infinitary disjunctions, which make it possible to characterize a number of constructions up to isomorphism by geometric structure and axioms. This gives rise to a geometric *mathematics*, going beyond the merely logical – technically it is the mathematics that can be conducted in the internal mathematics of toposes, and, in addition to that topos-validity, is moreover preserved by the inverse image functors of geometric morphisms. To put it another way, the geometric mathematics has an intrinsic continuity (since geometric morphisms are the continuous maps between toposes).

In this paper I shall survey some of the special features of geometric logic, and a body of established results that combine to support a manifesto "*continuity is geometricity*". In other words, to "do mathematics continuously" is to work within the geometricity constraints. In the rest of this section I shall set out four slogans of the manifesto, and the remaining sections will give the technical elaboration.

As one might expect, discussing continuity requires one first to discuss topological spaces, and the first slogan of the manifesto sets this out. It includes a rephrasing of Grothendieck's dictum that toposes are generalized topological spaces.

1. Spaces are geometric theories

To put this more carefully, a space is going to be described as the space of models for a geometric theory, with its topological nature arising naturally from that theory. This is in essence the approach of *point-free topology*, as adopted in locale theory and in formal topology, though we also generalize from propositional geometric theories to predicate ones, and thereby see Grothendieck's generalization from (point-free) topological spaces to toposes. There is ample evidence that it is the correct approach in a number of constructivist settings, including topos theory: point-free topology retains important results of classical topology that fail in a constructivist point-set approach.

If one stays with propositional geometric theories, the new spaces are equivalent to *locales* (see, e.g., [14,25]). One might say that they are *frames* (complete Heyting algebras) pretending to be topological spaces.

It is well known that there is an adjunction between topological spaces and locales, but to reach common ground (the *Stone* equivalence between sober spaces and spatial locales), concessions have to be made on both sides.

On the point-free side, the concession is to assume *spatiality*: that the frame can be embedded in a powerset (of a set of points). This is often thought of as harmless, since in classical mathematics enough important locales are spatial that the non-spatial ones can be regarded as pathological. This will not work constructively, however. Even for the real line, to validate standard results of analysis one needs a version (Examples 2 and 6) that is non-spatial in general. Thus in general it is essential to forgo spatiality and work outside the Stone equivalence.

On the point-set side, the concession is to assume that spaces are *sober*: that the assignment of open neighbourhood filters to points gives a bijection between points and completely prime filters of opens (or, which is classically equivalent, irreducible closed sets). This tells us that the points are not arbitrarily decreed as a set, but depend on some prior structure, the frame, and in fact the special features of sober topology carry over to point-free topology where the points are determined by the logical theory.

Any sober space is T_0, in other words each point is uniquely determined by its open neighbourhood filter. It also has the important property of being a *dcpo* (*directed complete poset*) with respect to the specialization order. It has all directed joins, found by taking directed unions of completely prime filters.

Accepting, as is inevitable in point-free topology, that the core of topology is sober, then the additional layer of non-sobriety in the usual theory can be understood as providing a set of labels (the arbitrarily decreed points) for some or all of the "abstract points" derived as a sober space from the topology. The labelling may have repetition, in other words the T_0 property may fail. The labelling can be described as a map from a discrete space to a sober space (or locale more generally), and as such is equivalent to a "topological system" as defined in [25]. However, we shall not be interested in such structures here. We are looking at the sober core of topology.

We can now discuss continuity. Note that, for us, the word *map* will always assume continuity.

2. Maps are point transformers, defined geometrically

In other words, a map $f : X \to Y$ is described by a geometric transformation $x \mapsto f(x)$.

There are two surprises here. The first is that geometric logic is incomplete, which means there may be an insufficiency of models to discriminate between logically inequivalent formulae. Traditionally one might see this as a deficiency in the logical rules, but in topos theory it is better seen as a deficiency in any individual set-theory's ability to supply models: this is the phenomenon of non-spatiality referred to above. For example, there are non-trivial locales with no points at all. Hence it is surprising that a map can be satisfactorily described as a point transformer. However, geometricity entails that the description can be applied not only to *global* points, maps $1 \to X$, of which there may be insufficient, but also to *generalized* points, maps $W \to X$ for arbitrary W, including the generic point $\mathrm{Id} : X \to X$. The global points are the models in the default category of sets (or base topos), while the generalized points allow the set theory to vary.

The second surprise is that no explicit continuity proof is required. Effectively, by adhering to geometricity constraints we forgo the ability to define discontinuous maps.