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A model of type theory in simplicial sets
A brief introduction to Voevodsky's homotopy type theory

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ABSTRACT

We describe how to interpret constructive type theory in the topos of simplicial sets where types appear as Kan complexes and families of types as Kan fibrations. Since Kan complexes may be understood as weak higher-dimensional groupoids this model generalizes and extends the (ordinary) groupoid model which was introduced by M. Hofmann and the author about 20 years ago. Finally, we discuss Voevodsky's Univalence Axiom which has been shown to hold in this model. This axiom roughly states that isomorphic types are equal. The type theoretic notion of isomorphism provided by this model coincides with homotopy equivalence of Kan complexes. For this reason it has become common to refer to it as Homotopy Type Theory.

1. Introduction

As observed in [6] identity types in intensional type theory endow every type with the structure of a *weak higher-dimensional groupoid*. The simplest and oldest notion of weak higher-dimensional groupoid is given by *Kan complexes* within the topos **sSet** of simplicial sets. This was observed around 2006 independently by V. Voevodsky and the author.

The aim of this note is to describe how simplicial sets organize into a model of Martin-Löf type theory. Moreover, we explain Voevodsky's *Univalence Axiom* which holds in this model and implements the idea that *isomorphic types are equal* as suggested in [6]. A full exposition of the theory will be given in a longer article by Voevodsky which is still in preparation, but see [10]. The current note just gives a first introduction to this circle of ideas.

2. Simplicial sets and Kan complexes

Due to limitation of space and time we can just give a very brief recap of this classical material (due to D. Kan and D. Quillen from late 1950s and 1960s). An excellent modern reference for this is the first chapter of [4].

Let $\mathbf{\Delta}$ be the category of finite nonempty ordinals and monotone maps between them. We write **sSet** for the topos $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$ of *simplicial sets*. We write $[n]$ for the ordinal $n+1 = \{0, 1, \ldots, n\}$ and $\Delta[n]$ for the corresponding representable object in **sSet**. For $0 \leqslant k \leqslant n$ we write $i^n_k : \Lambda_k[n] \hookrightarrow \Delta[n]$ for the inclusion of the k-th *horn* $\Lambda_k[n]$ into $\Delta[n]$ which is obtained by removing the interior and the face opposite to vertex k (for $n = 0$ the horn $\Lambda_0[0] = \Delta[0]$). There is an obvious[1] faithful functor $|\cdot|$ from $\mathbf{\Delta}$ into the category **Sp** of spaces and continuous maps. This induces the *singular* functor $\mathcal{S} : \mathbf{Sp} \to \mathbf{sSet}$ sending X to $\mathbf{Sp}(|-|, X)$ which has a left adjoint $\mathcal{R}$ called *geometric realization*. The objects in the image of $\mathcal{R}$ are the so-called CW-complexes which can be obtained by gluing simplices in a way as described by some simplicial set. The objects in the image of $\mathcal{S}$ are the so-called *Kan complexes* which can be characterized in a more combinatorial way as we will describe in the next paragraph.

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[1] With $[n]$ one associates the canonical n-dimensional simplex $\{x \in [0,1]^{n+1} \mid \sum x_i = 1\}$ endowed with the Euclidean topology. With $\alpha : [n] \to [m]$ one associates the continuous map $|\alpha|$ from the n-dimensional to the m-dimensional simplex defined as $|\alpha|(x)_j = \sum_{\alpha(i)=j} x_i$.

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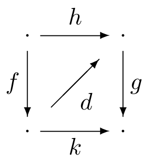
On **sSet** there is a well known *Quillen model structure* whose class $\mathcal{C}$ of *cofibrations* consists of all monos, whose class $\mathcal{W}$ of *weak equivalences* consists of all maps $f : X \to Y$ whose geometric realization $\mathcal{R}(f) : \mathcal{R}(X) \to \mathcal{R}(Y)$ is a homotopy equivalence[2] and whose class $\mathcal{F}$ of fibrations consists of all *Kan fibrations*, i.e. maps $a : A \to I$ in **sSet** with $i^n_k \perp a$ for all $k \leqslant n$ in $\mathbb{N}$. Here $f \perp g$ means that for every commuting square $kf = gh$ there is a (typically not unique) diagonal filler, i.e. a map d with $df = h$ and $gd = k$ as in



By definition a simplicial set X is a *Kan complex* iff $X \to 1$ is a Kan fibration. The extension property w.r.t. the horn inclusions $i^2_i : \Lambda_i[2] \hookrightarrow \Delta[2]$ expresses that "up to homotopy" morphisms can be composed and every morphism has a left and a right inverse.

It is shown in [4] that up to isomorphism Kan complexes are those simplicial sets which are isomorphic to $\mathcal{S}(X)$ for some space X. Moreover, one can show that a map $f : X \to Y$ between Kan complexes is a weak equivalence iff f is a homotopy equivalence, i.e. there is a map $g : Y \to X$ such that $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.[3]

In **sSet** one can develop a fair amount of homotopy theory and as shown in [4] inverting weak equivalences in **sSet** gives rise to the same *homotopy category* as inverting weak equivalences in **Sp**. Thus, from a homotopy point of view **sSet** and **Sp** are different ways of speaking about the same thing. However, the "combinatorial" topos **sSet** is in many respects much nicer then the "geometric" category **Sp**. This we will exploit when interpreting intensional Martin-Löf type theory in **sSet**.

3. Homotopy model for type theory

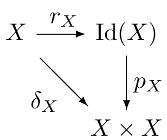
For basic information about type theory and its semantics see [5,8,9]. Type theory is the basis of interactive theorem provers like Coq as described in [2]. Since **sSet** is a topos and thus locally cartesian closed it provides a model of extensional type theory since **sSet** contains also a natural numbers object N.

In order to obtain a non-extensional interpretation of identity types we restrict families of types to be *Kan fibrations*. Accordingly, types are Kan complexes, i.e. weak higher-dimensional groupoids. In this respect the simplicial sets model appears as a natural generalization of the groupoid model of [6] which was our main motivation for introducing it.

Evidently, the class $\mathcal{F}$ contains all isomorphism and is closed under composition and pullbacks along arbitrary morphisms in **sSet**. Using the fact that trivial cofibrations are stable under pullbacks along Kan fibrations (referred to as *right proper* in the literature) one easily establishes that

Theorem 3.1. *Kan fibrations are closed under Π, i.e. whenever $a : A \to I$ and $b : B \to A$ are in $\mathcal{F}$ then $\Pi_a(b)$ is in $\mathcal{F}$, too.*

For interpreting equality on X we factor the diagonal $\delta_X : X \to X \times X$ as



with $p_X \in \mathcal{F}$ and $r_X \in \mathcal{C} \cap \mathcal{W}$ which is possible since $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a Quillen model structure. The Kan fibration p_X will serve as interpretation of

$$x, y\colon X \vdash \mathrm{Id}_X(x, y)$$

as suggested in [1].[4] For families of types as given by a Kan fibration $a : A \to I$ one factors the fibrewise diagonal $\delta_a : A \to A \times_I A$ in an analogous way. However, there is a problem since such factorizations are in general not stable under pullbacks. To overcome this problem we will introduce *universes* à la Martin-Löf.

As described in [10] a universe in **sSet** is a Kan fibration $p_U : \widetilde{U} \to U$. We write $\mathcal{D}_U$ for the class of Kan fibrations which can be obtained as pullbacks of p_U along some map in **sSet**. In [10] Voevodsky has shown how such a universe induces a contextual category $CC[p_U]$ which interprets dependent sums if $\mathcal{D}_U$ is closed under composition and which interprets dependent products if $\mathcal{D}_U$ is closed under Π.

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2 I.e. there exists a continuous map $g : \mathcal{R}(Y) \to \mathcal{R}(X)$ such that both composita are homotopy equivalent to the identities $\mathrm{id}_{\mathcal{R}(X)}$ and $\mathrm{id}_{\mathcal{R}(Y)}$, respectively.

3 For $f, g : A \to B$ we write $f \sim g$ iff there is a map $h : \Delta[1] \times A \to B$ with $h(0, -) = f$ and $h(1, -) = g$.

4 Already for ordinary groupoids (see [6]) the diagonal is hardly ever a fibration.