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Relational semantics for full linear logic

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ABSTRACT

Relational semantics, given by Kripke frames, play an essential role in the study of modal and intuitionistic logic. In [4] it is shown that the theory of relational semantics is also available in the more general setting of substructural logic, at least in an algebraic guise. Building on these ideas, in [5] a type of frames is described which generalise Kripke frames and provide semantics for substructural logics in a purely relational form.

In this paper we study full linear logic from an algebraic point of view. The main additional hurdle is the exponential. We analyse this operation algebraically and use canonical extensions to obtain relational semantics. Thus, we extend the work in [4,5] and use their approach to obtain relational semantics for full linear logic. Hereby we illustrate the strength of using canonical extension to retrieve relational semantics: it allows a modular and uniform treatment of additional operations and axioms.

Traditionally, so-called phase semantics are used as models for (provability in) linear logic [8]. These have the drawback that, contrary to our approach, they do not allow a modular treatment of additional axioms. However, the two approaches are related, as we will explain.

1. Introduction

Relational semantics, given by Kripke frames, play an essential role in the study of modal and intuitionistic logic [3]. They provide an intuitive interpretation of the logic and a means to obtain information about it. The possibility of applying semantical techniques to obtain information about a logic motivates the search for relational semantics in a more general setting.

Many logics are closely related to corresponding classes of algebraic structures which provide *algebraic semantics* for the logics. The algebras associated to classical modal logic are Boolean algebras with an additional operator (BAOs). Kripke frames arise naturally from the duality theory for these structures in the following way. Boolean algebras are dually equivalent to Stone spaces [11]. A modal operator on Boolean algebras translates to a binary relation with certain topological properties on the corresponding dual spaces, hence giving rise to so-called descriptive general frames. Forgetting the topology yields Kripke frames, which are in a discrete duality with perfect modal algebras, *i.e.*, modal algebras whose underlying Boolean algebra is a powerset algebra and whose operator is complete. This may be depicted as follows:

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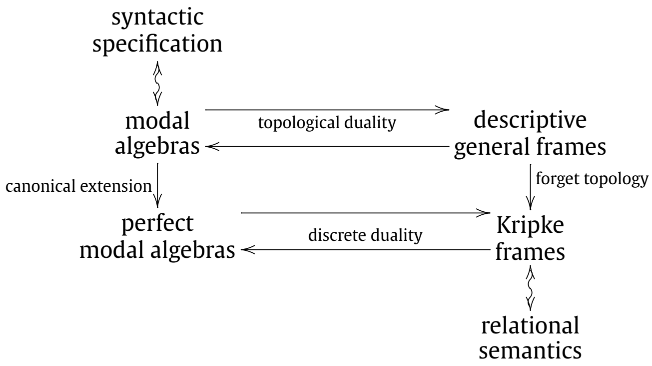
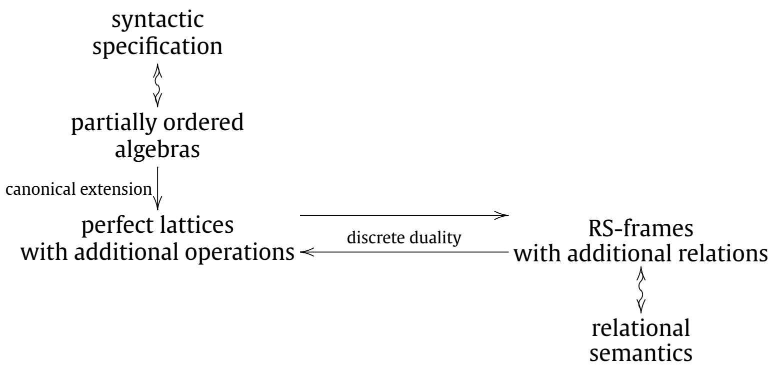


Hence, one may retrieve relational semantics for modal logic by first moving horizontally using the duality and thereafter going down by forgetting the topology.

Many other interesting logics, including substructural logics, however, have algebraic semantics which are not based on distributive lattices and for these duality theory is vastly more complicated or even non-existent. Luckily, the picture above also indicates an alternative route to obtain relational semantics: going down first and thereafter going right. The (left) downward mapping is given by taking the *canonical extension* of a BAO. Canonical extensions were introduced in the 1950s by Jónsson and Tarski exactly for BAOs [9,10]. Thereafter their ideas have been developed further, which has led to a smooth theory of canonical extensions applicable in a broad setting [6,7]. In [4] canonical extensions of partially ordered algebras are defined to obtain relational semantics for the implication–fusion fragment of various substructural logics. Their approach is purely algebraic. In [5] this work is translated to the setting of possible world semantics. A class of frames (RS-frames) is described which generalise Kripke frames and provide semantics for substructural logics in a purely relational form. This is summarised in the following picture:



(1)

A well-known substructural logic that extends the basic implication–fusion fragment is linear logic. Linear logic was introduced by Jean-Yves Girard [8]. Formulas in linear logic represent resources that may be used exactly once. Proof-theoretically this is witnessed by the fact that the structural rules contraction and weakening are not admissible in general. However, these structural rules are allowed in a controlled way by means of a new modality, the exponential !, which expresses the case of unlimited availability of a specific resource. Traditionally, phase spaces are used as semantics for linear logic. These have the drawback that, contrary to the approach described above, they do not allow a modular treatment of additional operations and axioms.

In [4] relational semantics for the basic implication–fusion fragment of linear logic was obtained. In this paper we extend this approach to derive relational semantics for full linear logic. We show that the axioms of the logic in question satisfy canonicity, and we identify the corresponding relational structures. We show that this method of canonicity and correspondence allows a modular and uniform treatment of the additional operations and axioms of linear logic. The modularity distinguishes our work from earlier derivations of Kripke-style semantics for linear logic [1]. Furthermore, we translate our results to one-sorted frames in order to compare these to phase semantics.

The paper is structured as follows: first, we discuss the general method of obtaining relational semantics for substructural logics using canonical extension, essentially by explaining how to move ‘down-right’ in the picture above (Section 2) and by indicating how to show that this indeed yields complete relational semantics (Section 3). We focus on the parts of this general theory that are important for the remainder of our paper and refer the reader to [4,5] for more details. In Section 4 this method is applied to obtain relational semantics for the multiplicative additive fragment of linear logic (MALL). The modular set-up allows us to augment this result by deriving relational semantics for the exponential, as we work out in Section 5. This gives relational semantics for full linear logic. In Section 6, which serves as an intermezzo, we look at the exponentials from the algebraic perspective. Finally, in Section 7 we discuss how our results relate to phase semantics.