Journal of Applied Logic 12 (2014) 192–207

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

Capturing equilibrium models in modal logic

Luis Fariñas del Cerro, Andreas Herzig, Ezgi Iraz Su *,[1]

Université de Toulouse, CNRS, IRIT, 118 Route de Narbonne, F-31062 Toulouse, Cedex 9, France

ARTICLE INFO

Article history:
Received 5 April 2013
Accepted 4 December 2013
Available online 7 February 2014

Keywords:
Equilibrium logic
Here-and-there models
Bimodal logic
Answer-set programming

ABSTRACT

Here-and-there models and equilibrium models were investigated as a semantical framework for answer-set programming by Pearce, Valverde, Cabalar, Lifschitz, Ferraris and others. The semantics of equilibrium logic is given in an indirect way: the notion of an equilibrium model is defined in terms of quantification over here-and-there models. We here give a direct semantics of equilibrium logic, stated for a modal language embedding the language of equilibrium logic.

1. Introduction

A here-and-there (HT) model (H,T) is a couple of sets of propositional variables, H ('here') and T ('there') such that $H \subseteq T$. We understand the inclusion informally as H being *weaker* than T. The logical language to talk about HT models has connectives $\bot$, $\wedge$, $\vee$, and $\rightarrow$. The latter is interpreted in a non-classical way and is therefore different from material implication $\supset$. Its truth condition is:

$$H,T \models \varphi \rightarrow \psi \quad \text{iff } H,T \models \varphi \supset \psi \quad \text{and} \quad T,T \models \varphi \supset \psi,$$

where $\supset$ is interpreted just as in classical propositional logic.[2] HT models give semantics to an implication with strength between intuitionistic and material implication. They were investigated by Pearce, Valverde, Cabalar, Lifschitz, Ferraris, and others as the basis of equilibrium logic, the latter providing a semantical framework for answer-set programming [20,19,22,5,6,14,18].

Equilibrium models of a formula, φ, are defined in an indirect way that is based on HT models: an equilibrium model of φ is a set of propositional variables T such that

* Corresponding author.
E-mail addresses: farinas@irit.fr (L. Fariñas del Cerro), herzig@irit.fr (A. Herzig), Ezgi-Iraz.Su@irit.fr (E.I. Su).

[1] Ezgi Iraz Su is not a member of CNRS.

[2] Material implication '$\supset$' here is just a shorthand enabling a concise formulation. To spell it out, its truth condition is: $H,T \models \varphi \supset \psi$ iff $H,T \not\models \varphi$ or $H,T \models \psi$.

http://dx.doi.org/10.1016/j.jal.2013.12.003
1570-8683/

1. $T \models \varphi$ in propositional logic, and
2. there is no HT model (H, T) such that H is strictly weaker than T and $H, T \models \varphi$.

Observe that the condition '$T \models \varphi$ in propositional logic' can be replaced by '$T, T \models \varphi$ in the logic of here-and-there'. To give an example, $T = \emptyset$ is an equilibrium model of $p \to \bot$ because (1) for the HT model $(\emptyset, \emptyset)$ we have $\emptyset, \emptyset \models p \to \bot$, and (2) there is no set H that is strictly included in the empty set. Moreover, $T = \emptyset$ is the only equilibrium model of $p \to \bot$. To see this, suppose T is an equilibrium model for $p \to \bot$ for some $T \neq \emptyset$. Then T cannot contain p, otherwise condition (1) would be violated. Therefore T contains q for some $q \neq p$, but then condition (2) is violated since $\emptyset, T \models p \to \bot$.

In the present paper we give a direct semantics of equilibrium logic in terms of a modal language extending that of propositional logic by two unary modal operators, [T] and [S]. Roughly speaking, [T] allows to talk about valuations[3] that are at least as strong as the actual valuation; and [S] allows to talk about valuations that are weaker than the actual valuation. Our modal language can be interpreted on HT models. However, we also give a semantics in terms of Kripke models. We call our logic **MEM**: the Modal Logic of Equilibrium Models.

We relate the language of equilibrium logic to our bimodal language by means of the Gödel translation, *tr*, whose main clause is:

$$tr(\varphi \to \psi) = [\mathrm{T}]\big(tr(\varphi) \supset tr(\psi)\big).$$

A first attempt to relate equilibrium logic to modal logic in the style of the present approach was presented in [12]. We here extend and improve that paper by simplifying the translation.

The paper is organised as follows. In Section 2 we introduce our modal logic of equilibrium models, **MEM**,[4] syntactically, semantically and also axiomatically. In Section 3 we recall both the logic of here-and-there and equilibrium logic. In Section 4 we define the Gödel translation, *tr*, from the language of the logic of here-and-there to the language of **MEM** and prove its correctness: for every formula φ, φ is HT valid if and only if $tr(\varphi)$ is **MEM** valid. This theorem paves the way for the proof of the grand finale given in Section 5: φ is a logical consequence of χ in equilibrium logic if and only if the modal formula

$$\big(tr(\chi) \wedge [\mathrm{S}]\neg tr(\chi)\big) \supset tr(\varphi)$$

is valid in **MEM**. It follows that φ has an equilibrium model if and only if $tr(\varphi) \wedge [\mathrm{S}]\neg tr(\varphi)$ is satisfiable in the corresponding Kripke model. Section 6 makes a brief overview of our past, present and future interests. They all appear in a line of work that aims to reexamine the logical foundations of equilibrium logic and answer-set programming.

2. The modal logic of equilibrium models: MEM

We introduce the modal logic of equilibrium models, **MEM**, in the classical way: we start by defining its bimodal language and its semantics. Then we axiomatise its validities.

2.1. Language

Throughout the paper we suppose $\mathbb{P}$ is a countably infinite set of propositional variables. The elements of $\mathbb{P}$ are noted p, q, etc. Our language $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$ is bimodal: it has two modal operators, [T] and [S]. Precisely, $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$ is defined by the following grammar:

[3] Here, and in general in this paragraph, we use the term 'valuation' in the sense of a set of proposition variables.

[4] To avoid confusion we could have used another name instead of **MEM** again. It should however be clear to the reader that the modal logic we are talking about, here, is just slightly different from the one we introduced in [12].