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A concept approach to input/output logic

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This paper provides a semantics for input/input output logic based on formal concept analysis. The central result shows that an input/output logic axiomatised by a relation R is the same as the logic induced by deriving pairs from the concept lattice generated by R using a $\wedge$- and $\vee$-classical Scott consequence relation. This correspondence offers powerful analytical techniques for classifying, visualising and analysing input/output relations, revealing implicit hierarchical structure and/or natural clusterings and dependencies. The application of all formal developments are illustrated by a worked example towards the end.

1. Introduction

Input/output logic is a branch of conditional logic, broadly conceived, whose distinguishing feature is that it does not make any assumptions about the ultimate nature of the relation that holds between a set of conditions and its consequences. That may not sound like much, but it makes a real difference, both formally and philosophically.

Philosophically, it goes against a habit of logicians of assimilating all kinds of connections between a condition and a consequence to the inference paradigm: causality becomes causal *inference*, by which is meant the drawing of conclusions about a causal connection based on the conditions for the occurrence of an effect. The study of sets of norms becomes the study of normative *reasoning*, by which is usually meant the application of a practical syllogism or the drawing of a conclusion about the optimality of some state of affairs.

Although this method of investigation is sometimes natural, it is certainly not inevitable, nor is it always the most direct strategy available. After all, studying causality by way of our reasoning about it is a somewhat roundabout way of approaching the object. The causal relation itself is not an inference relation, strictly speaking, it is a relation between *things* or natural phenomena. At best, co-variation of causes and effects correspond to conditionals only in a derivative sense therefore. Analogical remarks apply to sets of norms: a norm is not primarily a conditional, it is a stipulation that holds by decree.

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Input/output logic gives expression to this latter way of looking at things. A correlation between a condition and a consequence is seen as just an element in an ordinary binary relation between states of affairs as described by formulae. The ultimate nature and origin of this relation is left open, which just means that input/output logic does not foreclose any interpretational options.

On a formal level, the thrust of this general stance—which is justified in more detail in [10]—is to shift the emphasis from a theory formulated in terms of the behaviour of object-language connectives to a theory formulated in terms of the behaviour of sets and relations. Its methodological significance consists in the fact that it allots to philosophical logic a parcel in a wider mathematical landscape where logic is naturally tangent upon e.g. lattice theory and universal algebra.

The present discussion proceeds subject to this general conception to study the particular tangential point which exists between input/output logic and the branch of lattice theory called formal concept analysis (FCA for short). The basic idea is this: formal concept lattices offer powerful, well-studied, analytical techniques for classifying, visualising and analysing binary relations, revealing implicit hierarchical structure and/or natural clusterings and dependencies between the objects of the relation [5]. Since the set of axioms in any given input/output logic is just a binary relation between formulae, it ought to be possible to apply results from FCA to the study of forms of conditionality that are not naturally assimilated to the model based on inference relations and/or conditionals—e.g. to sets of norms (an example that will be used for illustration purposes throughout the paper). This idea was first proposed in [19], but without being developed in much detail.

The principal research question of the present paper is the following: given a binary relation G over sets of formulae, what is the relation between the concept lattice induced by G on the one hand and the input/output logic *axiomatised* by G on the other? The paper does not answer this question generally, but limits itself to a couple of more specific results summarised below:

First, there is a faithful embedding into FCA of the system of input/output logic called *basic output* in the nomenclature of Makinson and van der Torre. More precisely, we have that A) the concept lattice induced by the relation G is equivalent (in a sense to be made precise) to G itself modulo basic output, and B) there is a rule for evaluating any pair of formulae (a, b) against the concept lattice induced by G that answers yes iff (a, b) is in system of basic output axiomatised by G. Stated differently, for any given system of basic output, the concept lattice induced by the axioms of that system can A) itself be turned into a set of axioms for the system, and B) constitutes a semantic structure for it.

The second result concerns the system of input/output logic that Makinson and van der Torre call *basic reusable output*, which is the system that results from adding a rule of cumulative transitivity to the set of rules for basic output. The theme is now varied a bit: whereas it would be convenient to try and extend the results for basic output in the straightforward manner and ask whether the concept lattice induced by G can be turned into a set of axioms for the system of basic *reusable* output having G as a set of axioms, this would not be a very interesting exercise given the goals of the present paper. The central concern of the present paper is to bring input/output logic into the ambit of formal concept analysis, more specifically to make it possible to draw on lattice theoretic techniques in the analysis of a given input/output system. Obviously, this requires information to be encoded in the concept lattice. Yet, if one were to extend the results for basic output in the straightforward manner, then the set of generators/axioms of the logic would remain the same and so would the corresponding concept lattice. In other words, the concept lattice would not reflect the change of logic.

Instead therefore, the strategy that chosen is to encode pairs whose derivation is licensed by the rule of cumulative transitivity into the concept lattice itself by extending the relation G to a larger relation G^+ that still induces a finite lattice. This process is referred to as *saturating* the axioms—or simply as saturation. It is proved that saturation yields an axiomatisation for a system of basic reusable output analogous to A) above, although the procedure is not exhaustive enough to supply a *non-trivial* semantics in the sense of B).