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Topological FL_{ew}-algebras

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ABSTRACT

The main goal of this article is to introduce topological FL_{ew}-algebras and study their main properties. We also treat completions of FL_{ew}-algebras with respect to inductive family of filters. This work generalizes similar works on MV-algebras [10] and on FL_{ew}-algebras equipped with uniform topologies [9].

1. Introduction

Universal topological algebras (which are algebras with topologies with respect to which all operations are continuous) have been introduced and studied thoroughly (see e.g. [4,12,13,19]). Some of the notable classes of topological algebras that have been the objects of more detailed studies include groups [17], lattices [16], orthomodular lattices [5], MV-algebras [10].

A residuated lattice is a lattice equipped with a monotone monoidal operation $\odot$ (with a unit e) and a pair of binary operations $/, \backslash$ satisfying

$$x \odot y \leq z \quad \text{if and only if} \quad x \leq z/y \quad \text{if and only if} \quad y \leq x\backslash z$$

FL-algebras, which are residuated lattices expanded by a constant $\bar{0}$ [15], form the algebraic semantics for the so-called (intuitionistic) substructural logics. These provide a unifying framework for several kinds of logics (Girard's linear logic, the Lambek calculus, Łukasiewicz's many-valued logics, the Hájek fuzzy logics etc.). A very important subclass of FL-algebras is that of FL_{ew}-algebras, which are FL-algebras extended by the

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exchange and weakening rules. An $\mathrm{FL_{ew}}$-algebra (also known as bounded integral commutative residuated lattice) can be defined [7,11] as a bounded lattice $(L, \vee, \wedge, 0, 1)$ with two additional binary operations $\odot, \rightarrow$ satisfying: (i) $(L, \odot, 1)$ is a commutative monoid, and (ii) for all $x, y, z \in L$, $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$. A nonempty subset F of an $\mathrm{FL_{ew}}$-algebra L is called a filter of L if it satisfies: (F1) $x \odot y \in F$ for all $x, y \in F$, and (F2) For all $x, y \in L$, if $x \leq y$ and $x \in F$, then $y \in F$. While, filters of an $\mathrm{FL_{ew}}$-algebra are studied for their connection to its congruences, they primarily come from logic itself since they correspond to theories of the logic. Indeed, filters of an $\mathrm{FL_{ew}}$-algebra L are just subsets of L that are closed under all deductions. Equivalently, in algebraic logic, filters are sets of designated elements that provide matricial models for the logic.

Among the many important subclasses of $\mathrm{FL_{ew}}$-algebras, there is that of MV-algebras, which constitute the algebraic counterpart of Łukasiewicz many valued logic. In [10], Hoo introduced topological MV-algebras, and studied their main properties. A close analysis of Hoo's work reveals that the essential ingredients are the existence of an adjoint pair of operations and the fact that ideals of MV-algebras correspond to its congruences. This prompted us to consider the same study in a more general context where similar ingredients are available, namely $\mathrm{FL_{ew}}$-algebras. The main goal of the present work is to generalize Hoo's work to $\mathrm{FL_{ew}}$-algebras. This yields the notion of topological $\mathrm{FL_{ew}}$-algebra, which is an $\mathrm{FL_{ew}}$-algebra together with topology with respect to which all the operations are continuous. Topological $\mathrm{FL_{ew}}$-algebras were already investigated in [9], but mainly for the uniform topology.

The paper is organized as follows. In Section 2, we study the general properties of topological $\mathrm{FL_{ew}}$-algebras. In Section 3, we study a special class of topological $\mathrm{FL_{ew}}$-algebras, namely those arising from decreasing families of filters indexed by directed sets. We establish conditions under which the resulting space has certain topological properties, and also study completions of topological $\mathrm{FL_{ew}}$-algebras. In addition, we show that such $\mathrm{FL_{ew}}$-algebras are Stone if and only of they are compact and Hausdorff.

Homomorphisms of $\mathrm{FL_{ew}}$-algebras have the usual meaning.

A subset F of an $\mathrm{FL_{ew}}$-algebras L is called a deductive system of L if:

$$\text{(ds1) } 1 \in F \text{ and (ds2) For every } x, y \in L, \text{ if } x, x \rightarrow y \in F, \text{ then } y \in F.$$

It is known that the notions of filters and deductive systems coincide (see e.g., [8]). We shall use solely the filter terminology in the entire article.

Filters of $\mathrm{FL_{ew}}$-algebras induce congruences, indeed the following result is well-known.

Proposition 1.1. *(See [18].) Let L be an $\mathrm{FL_{ew}}$-algebra and let F be a filter of L. The relation $x \equiv_F y$ if and only if $x \rightarrow y, y \rightarrow x \in F$ is a congruence on L.*

If F is a filter of L, the quotient $\mathrm{FL_{ew}}$-algebra induced by the congruence $\equiv_F$ shall be denoted by L/F. In addition the class of an element $a \in L$ with respect to $\equiv_F$ is often denoted by $[a]_F$; and π_F denotes the natural projection $L \rightarrow L/F$.

For every subset $X \subseteq L$, the smallest filter of L containing X (i.e., the intersection of all filters F of L such that $X \subseteq F$) is called the filter generated by X and will be denoted by $\langle X \rangle$.

If U and V are subsets of L and $\star \in \{\vee, \wedge, \rightarrow, \odot\}$, then

$$U \star V := \{x \star y : x \in U, y \in V\}$$

Remark 1.2. While some aspects of the paper could be treated in the more general context of FL-algebras, or even arbitrary residuated lattices, we restrict our study to bounded integral commutative residuated lattices ($\mathrm{FL_{ew}}$-algebras). Our choice is mainly motivated by the fact that the definitions and notations in the $\mathrm{FL_{ew}}$-algebras context are greatly simplified, and that once established, one could easily adapt the relevant results to that more general context.