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Gödel spaces and perfect MV-algebras

Antonio Di Nola [a], Revaz Grigolia [b,*]

[a] University of Salerno, Via Giovanni Paolo II, 132, 84084 Fisciano (SA), Italy
[b] Tbilisi State University, University Str., 13, 0147 Tbilisi, Georgia

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ABSTRACT

The category of Gödel spaces $\mathcal{GS}$ (with strongly isotone maps as morphisms), which are dually equivalent to the category of Gödel algebras, is transferred by a contravariant functor $\mathcal{H}$ into the category $\mathbf{MV(C)^G}$ of MV-algebras generated by perfect MV-chains via the operators of direct products, subalgebras and direct limits. Conversely, the category $\mathbf{MV(C)^G}$ is transferred into the category $\mathcal{GS}$ by means of a contravariant functor $\mathcal{P}$. Moreover, it is shown that the functor $\mathcal{H}$ is faithful, the functor $\mathcal{P}$ is full and the both functors are dense. The description of finite coproduct of algebras, which are isomorphic to Chang algebra, is given. Using duality a characterization of projective algebras in $\mathbf{MV(C)^G}$ is given.
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1. Introduction

MV-algebras are the algebraic counterpart of the infinite valued Łukasiewicz sentential calculus, as Boolean algebras are with respect to the classical propositional logic. In contrast to what happens for Boolean algebras, there are MV-algebras which are not semisimple, i.e. the intersection of their maximal ideals (the radical of A) is different from $\{0\}$. Non-zero elements from the radical of A are called infinitesimals. Perfect MV-algebras are those MV-algebras generated by their infinitesimal elements or, equivalently, generated by their radical [3]. Let L_P be the logic of perfect MV-algebras that coincides with the set of all Łukasiewicz formulas that are valid in all perfect MV-chains, see [3].

As it is well known, MV-algebras form a category that is equivalent to the category of abelian lattice ordered groups (ℓ-groups, for short) with strong unit [22]. Let us denote by Γ the functor implementing this equivalence. If G is an ℓ-group, then for any element $u \in G$, $u > 0$ we let $[0, u] = \{x \in G : 0 \leq x \leq u\}$ and for each $x, y \in [0, u]$ $x \oplus y = u \wedge (x + y)$ and $\neg x = u - x$. In particular each perfect MV-algebra is associated with an abelian ℓ-group with a strong unit. Moreover, the category of perfect MV-algebras is equivalent to the category of abelian ℓ-groups, see [14].

* Corresponding author.
E-mail addresses: adinola@unisa.it (A. Di Nola), revaz.grigolia@tsu.ge (R. Grigolia).

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The class of perfect MV-algebras does not form a variety and contains non-simple subdirectly irreducible MV-algebras. It is worth stressing that the variety generated by all perfect MV-algebras, denoted by $\mathbf{MV}(\mathbf{C})$, is also generated by a single MV-chain, actually the MV-algebra C, defined by Chang in [6]. We name *$MV(C)$-algebras* all the algebras from the variety generated by C. Notice that the Lindenbaum algebra of L_P is an $MV(C)$-algebra. The perfect algebra C has relevant properties. Indeed C generates the smallest variety of MV-algebras containing non-boolean non-semisimple algebras. It is also subalgebra of any non-boolean perfect MV-algebra.

The importance of the class of $MV(C)$-algebras and the logic L_P can be perceived by looking further at the role that infinitesimals play in MV-algebras and Łukasiewicz logic. Indeed the pure first order Łukasiewicz predicate logic is not complete with respect to the canonical set of truth values $[0, 1]$, see [27,2]. The Lindenbaum algebra of the first order Łukasiewicz logic is not semisimple and the valid but unprovable formulas are precisely the formulas whose negations determine the radical of the Lindenbaum algebra, that is the co-infinitesimals of such algebra. Hence, the valid but unprovable formulas generate the perfect skeleton of the Lindenbaum algebra. So, perfect MV-algebras, the variety generated by them and their logic are intimately related with a crucial phenomenon of the first order Łukasiewicz logic.

Let us introduce some notations: let $C_0 = \Gamma(Z, 1)$, $C_1 = C \cong \Gamma(Z \times_{lex} Z, (1, 0))$ with generator $(0, 1) = c_1 (= c)$, $C_m = \Gamma(Z \times_{lex} \cdots \times_{lex} Z, (1, 0, \ldots, 0))$ with generators $c_1(= (0, 0, \ldots, 1)), \ldots, c_m(= (0, 1, \ldots, 0))$, where the number of factors Z is equal to $m \geq 1$ and $\times_{lex}$ is the lexicographic product. Let us denote $Rad(A) \cup \neg Rad(A)$ through $R^*(A)$.

We are interested in the class $\mathbf{LSP}\{C_i : i \in \omega\}$ of $MV(C)$-algebras which is generated by the set $\{C_i : i \in \omega\}$ by the operators of direct products, subalgebras and direct limits, where C_0 is two-element Boolean algebra, $C_1 = C$ and C_n $(n > 1)$ is the n-generated perfect MV-chain.

Let $\mathbf{K}$ be any variety of algebras. Then $F_{\mathbf{K}}(m)$ denotes the m-generated free algebra in the variety $\mathbf{K}$.

Now we introduce the notion of *weak duality* between categories. Let $\mathbf{A}, \mathbf{B}$ be categories. We say that $\mathbf{A}$ and $\mathbf{B}$ are *weakly dual* (or that there is a *weak duality* between $\mathbf{A}$ and $\mathbf{B}$) if there are dense contravariant functors $F_{\mathbf{A}} : \mathbf{A} \to \mathbf{B}$ and $F_{\mathbf{B}} : \mathbf{B} \to \mathbf{A}$ such that $F_{\mathbf{A}}$ is faithful and $F_{\mathbf{B}}$ is full.

In this paper, we give the description of m-generated free algebras in the variety $\mathbf{MV}(\mathbf{C})$ generated by perfect MV-algebras. We describe the category of Gödel spaces, where any Gödel space is a special case of Priestley spaces. We also will prove that there is a weak duality between the full subcategory $\mathbf{MV}(\mathbf{C})^{\mathbf{G}}(= \mathbf{LSP}\{C_i : i \in \omega\})$ of the category $\mathbf{MV}(\mathbf{C})$ and the category of Gödel spaces $\mathcal{GS}$. More precisely, we construct the functors $\mathcal{P} : \mathbf{MV}(\mathbf{C})^{\mathbf{G}} \to \mathcal{GS}$, which is full, and $\mathcal{H} : \mathcal{GS} \to \mathbf{MV}(\mathbf{C})^{\mathbf{G}}$ which is faithful.

In the category theory, a functor $\mathcal{F} : \mathbf{E} \to \mathbf{D}$ is *dense* (or *essentially surjective*) if each object D of $\mathbf{D}$ is isomorphic to an object of the form $\mathcal{F}(E)$ for some object E of $\mathbf{E}$. The suggested functors $\mathcal{P} : \mathbf{MV}(\mathbf{C})^{\mathbf{G}} \to \mathcal{GS}$ and $\mathcal{H} : \mathcal{GS} \to \mathbf{MV}(\mathbf{C})^{\mathbf{G}}$ are dense.

The category $\mathcal{GS}$ of Gödel spaces is dually equivalent to the category $\mathbf{GA}$ of Gödel algebras. Hence, there exist two functors $\mathcal{G} : \mathbf{GA} \to \mathcal{GS}$ and $\mathcal{HS} : \mathcal{GS} \to \mathbf{GA}$. So, we also have two functors $\mathcal{HS} \circ \mathcal{P} : \mathbf{MV}(\mathbf{C})^{\mathbf{G}} \to \mathbf{GA}$ and $\mathcal{H} \circ \mathcal{G} : \mathbf{GA} \to \mathbf{MV}(\mathbf{C})^{\mathbf{G}}$. Moreover, $\mathcal{HS} \circ \mathcal{P}$ coincides with the Belluce functor β [1] defined on the $\mathbf{MV}(\mathbf{C})^{\mathbf{G}}$.

Using the weak duality we give a construction of a coproduct in $\mathbf{MV}(\mathbf{C})^{\mathbf{G}}$ which coincides with coproduct in $\mathbf{MV}(\mathbf{C})$. Moreover, we show that the coproduct coincides with free product (using this weak duality). Free products in various classes of ℓ-groups were investigated in the frame of varieties of ℓ-groups or abelian ℓ-groups by Holland and Scrimger [17], Martinez [20,21], Powel and Tsinakis [25], Mundici [23], Dvurečenskij and Holland [15], Di Nola and Lettieri [14]. Moreover, D. Mundici in [23] has shown that coproduct coincides with free product in the variety of MV-algebras.

We notice that in [10] it is established a duality between the category of finitely generated $MV(C)$-algebras, having finite spectrum, and the category of finite dual Heyting algebras which satisfy linearity condition.

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